DOI: 10.13656/j.cnki.gxkx.1996.03.002
广西科学 Guangxi Sciences 1996，3（3）：4～7

The Estimation of Lower Bounds about Some Ramsey Number $R_n(3)$ and $R_n(4)$
若干 Ramsey 数 $R_n(3)$和 $R_n(4)$的下界估计

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Abstract The basic character of prime number order cycle graph was studied by the method of construction，and the new lower bounds about some Ramsey numbers were obtained.

Key words cyclic graph，linear transformation，Ramsey number

摘要 用构造性的方法研究了素数阶循环图的基本性质，得到若干 Ramsey 数的新的下界。

关键词 循环图 线性变换 Ramsey 数

It is a famous difficulty to determine Ramsey number in combination mathematics and graph theory (see [1～9]). The comprehensive document [10] listed that the accurate value and the upper and lower bounds of some Ramsey numbers have been known currently. The basic character of prime number order cycle graph was studied by the method of construction，and the new lower bounds about some Ramsey numbers were obtained.

Theorem 1 Convention，for the Ramsey number $R(k_1,k_2,\cdots,k_n;2)$ ，it is written simply as $R_n(k)$ when $k_1=k_2=\cdots=k_n=k$. We have

Ⅰ. $R_5(3)\geqslant 102$，　$R_6(3)\geqslant 278$，
$R_7(3)\geqslant 492$，　$R_9(3)\geqslant 578$，
$R_{10}(3)\geqslant 1182$.

Ⅱ. $R_3(4)\geqslant 128$，　$R_4(4)\geqslant 458$，
$R_5(4)\geqslant 942$，　$R_6(4)\geqslant 3458$，
$R_8(4)\geqslant 9698$，　$R_{10}(4)\geqslant 17682$，
$R_{12}(4)\geqslant 28298$，　$R_{14}(4)\geqslant 47798$，
$R_{20}(4)\geqslant 84962$，　$R_{22}(4)\geqslant 87870$，
$R_{24}(4)\geqslant 155378$，　$R_{28}(4)\geqslant 207482$，
$R_{29}(4)\geqslant 230552$，　$R_{30}(4)\geqslant 287702$，
$R_{32}(4)\geqslant 345090$.

The technical terms of numbet Theory，Group Theory，Graph Theory quoted in the paper can be seen in references [11，12，1].

1 Linear transformation of cyclic graph

Set integer number $n\geqslant 2$ ，prime number $p=2mn+1$ ，note $Z_p=\{-mn,\cdots,-1,0,1,\cdots,mn\}$ as the minimum absolute value's complete system of residues of module p ，convention，in the following，except special decalaring，all small English letters refer to module p integer number，and results of the operation of any integer number's addition，subtraction，multiplication，division（for simplicity and convenience，we still use "="）must be taken module p coresidual to Z_p. Assume g is primitive root of p. note

$$\overline{Z}_p=\{x\mid x=g^j,0\leqslant j<2mn\}$$
$$\alpha_i=\{x\mid x=g^{nj+i},0\leqslant j<2m\}.\quad 0\leqslant i<n$$
$$\alpha_i\alpha_j=\{x\mid x=ab,a\in\alpha_i,b\in\alpha_j\}$$

As we know，Z_p is finite field，$\overline{Z}_p$ is $2mn$ orders commutative group in the operation of multiplication of module p coresidual. α_0 is $2m$ orders cyclic groups whose generating element is g^n. It is normal subgroup of $\overline{Z}_p$. α_i is cosets of α_0，

$$\overline{Z}_p/\alpha_0=\{\alpha_0,\cdots,\alpha_{n-1}\}$$

1996-06-10 收稿。

is quotient groups of $\overline{Z}_p$, and has:

Proposition 1 $\alpha_i\alpha_j = \alpha_{i+j}$, here $\alpha_{i+j} = \alpha_r$, $r \equiv i + j \pmod n$ and $0 \leqslant r < n$ (convention, in the following, the subscript about α_i all model these: make module n coresidual and sum up to the min nonegative system of residues of module n).

Definition 1 Assume G is complete graph of p vertexes, the set of vertex $V_G = Z_P$, the set of side $E_G = \{\alpha_0, \cdots, \alpha_{n-1}\}$ (that is to say, the colores named as $0, 1, \cdots, n-1$ make colouring every edge): If and only if $x - y \in \alpha_i$, the two vertexes X, Y, are called as α_i adjacency (the united side of the vertex X and Y is coloured i), the complete graph that is named every vertex and provided the method of every edge's colouring is called P orders cyclic graphs.

Definition 2 In p orders cyclic graphs G. $k \geqslant 2$ different vertexes $x, y, \cdots, z$, if any two vertexes are α_i adjacency, we say that they make a k orders α_i cligue (k orders complete subgraph of every side coloured i). Note them $(x, y, \cdots, z)_i$. When they wouldn't be misunderstood, we omit α_i cligue's subscript, note them simply as $(x, y, \cdots, z)$. $x, y, \cdots, z$ are called elements of the cligue. The two cligues having the same element (Whether their element sequence is the same or not) are considered as one cligue and have not any difference.

Definition 3 Giving two p orders cyclic graph G and G'. If there is monogamy relation f between V_G and $V_{G'}$, and f map α_i cligue in graph G onto α_j cligue in graph G'. Then the two graphs are called isomorphic. Convention, two isomorphic p orders cyclic graphs (all of their vertex graphs are Z_p) are considered as one graph. Isomorphic map f stated above is called transformation of graph G.

Proposition 2 Assume $a \in \alpha_j$, $b \in Z_p$, thus $f(x) = ax + b$ ($x \in Z_p, f(x) \in Z_p$) makes transformation (linear transformation) of graph G. It transforms k orders α_i cligue to k orders α_{i+j} cligue.

Proof Notice $a \in \alpha_j \Rightarrow a \neq 0$, for any $x, y \in Z_p$, we have

$$f(x) = f(y) \Leftrightarrow a(x - y) = 0 \Leftrightarrow x = y.$$

That is to say that f make $1-1$ transformation of vertex sets V_G. According to Proposition 1, we have

$$x - y \in \alpha_i \Leftrightarrow f(x) - f(y) = a(x - y) \in \alpha_{i+j}$$

That is to say that two adjacent α_i apexes in graph G transform other two α_{i+j} adjacent α_i apexes. So f transform α_i cligue in gralph G to the same orders α_{i+j} cligue. Proof is over.

Proposition 3 If transformation f transforms k orders α_i cligue $(x_1, x_2, \cdots, x_k)_i$ to k orders α_j cligue $(y_1, y_2, \cdots, y_k)_j$, we note:

$$f(x_1, x_2, \cdots, x_k)_i = (y_1, y_2, \cdots, y_k)_j.$$

here, $y_t(x) = f(x_t)$, $1 \leqslant t \leqslant k$. Thus for transformations

$$f_1(x) = (x_2 - x_1)^{-1}(x - x_1).$$

and $f_2(x) = 1 - x$, we have:

$$f_1(x_1, x_2, \cdots, x_k)_i = (0, 1, \cdots, y_k)_0 \qquad (1).$$

$$f_2(0, 1, \cdots, y_k)_0 = (0, 1, \cdots, 1 - y_k)_0 \qquad (2).$$

Proof From Difinition 2 we know $x_2 - x_1 \in \alpha_i$, from Proposition 1 we know $(x_2 - x_1)^{-1} \in \alpha_{-i}$, from Proposition 2 we know f_1 transform α_i cligue to α_0 cligue and formula (1). Noticing that α_0 is $2m$ orders cyclic group whose generating element is g^n. So $g^{nm} \neq 1$, but $(g^{nm})^2 = 1$, we can get $g^{nm} = -1 \in \alpha_0$, from Proposition 2 we know that $f_2(x) = (-1) \cdot x + 1$ transforms α_0 cligue to α_0 cligue and get formula (2). Proof is over.

2 Normal Subgroup α_0 and Lower Bounds of $R_n(3)$ and $R_n(4)$

As we all know, there is a famous theorem in graph theory——Ramsey Theorem: For any $n \geqslant 2$ positive ingegers: $k_1, k_2, \cdots, k_n \geqslant 2$, there is the minimum positive integer R, when $S \geqslant R$, we make the side of S orders complete graph G any colouring with n kinds of colors. Then there must exist k_i orders complete subgraph whose every edge is coloured with the same No. i color. Here i is one of 1, 2, $\cdots$, n.

Positive integer R stated above is called Ramsey number $R(k_1, k_2, \cdots, k_n; 2)$. When $k_1 = k_2 = \cdots = k_n = k$, we simply note it as $R_n(k)$ and we have:

Theorem 2 In p orders cyclic graph G, note the positive element of α_0 as generator subgroup:

$$\alpha_0^+ = \{x \mid x \in \alpha_0 \text{ and } x > 0\}.$$

If for any $x \in \alpha_0^+$, $x - 1 \overline{\in} \alpha_0^+$ for ever, then $R_n(3) \geqslant p + 1$.

Proof On the condition of Theorem 2, we prove that there doesn't exist any a certain three orders α_i cligue. Otherwise, assume there exits a three orders α_i

cligue (x_1, x_2, x_3), according to Proposition 3, we know

$$f_1(x_1, x_2, x_3)_i = (0,1,a)_0.$$

$$f_2(0,1,a)_0 = (0,1,1-a)_0.$$

here $a = f_1(x_3)$. According to Definition 2, we know a, $a-1, 1-a, (1-a)-1 \in \alpha_0$. Thus when $a \in \alpha_0^+$, we can get $a-1 \in \alpha_0^+$ or when $a \overline{\in} \alpha_0^+$, we have $1-a \in \alpha_0^+$ and $(1-a)-1 = -a \in \alpha_0^+$. The two kinds of results are in contradiction with the condition of Theorem 2. So we can prove that there doesn't exist any 3 orders α_i cligue. According to Ramsey Theorem we know $R_n(3) \leqslant p$ is impossible. So we can get $R_n(3) \geqslant p+1$. Proof is over.

Theorem 3 In p orders cyclic graph G, we note subset of α_0 as:

$$\theta = \{x \mid x \in \alpha_0 \text{ and } x-1 \in \alpha_0\}$$

$$\theta^+ = \{x \mid x \in \alpha_0^+ \text{ and } x-1 \in \alpha_0^+\}$$

Assume $\theta \neq \varnothing$, $a \in \theta$, we order

$$\theta(a) = \{x \mid x \in \theta \text{ and } x-a \in \alpha_0\}$$

If for any $a \in \theta^+$, $\theta(a) = \varnothing$ for ever, thus $R_n(4) \geqslant p+1$.

Proof Assume that there is a certain 4 orders α_i cligue (x_1, x_2, x_3, x_4), from Proposition 3 we know

$$f_1(x_1, x_2, x_3, x_4)_i = (0,1,a,b)_0.$$

$$f_2(0,1,a,b)_0 = (0,1,1-a,1-b)_0.$$

Here $a = f_1(x_3)$, $b = f_1(x_4)$. According to Definition 2 we know $a, b, a-1, b-1, -a, -b, 1-a, 1-b, b-a \in \alpha_0$. So $a, b, 1-a, 1-b \in \theta$ and $b \in \theta(a) \neq \varnothing$, $1-b \in \theta(1-a) \neq \varnothing$. No matter that $a \in \theta^+$, or $a \overline{\in} \theta^+$, that is to say: $1-a \in \theta^+$. The two situations are in contradiction with the condition of Theorem 3.

Then we prove that on the condition of Theorem 3, there doesn't exist any 4 orders α_i cligue. From Ramsey Theorem, we know $R_n(4) \leqslant p$ is impossible, and there is only $R_n(4) \geqslant p+1$. Proof is over.

Because normal subgroup α_0 and its subset is an important role on the lower bound's estimation of $R_n(3)$ and $R_n(4)$, We initially study their structure.

Proposition 4 $\alpha_0 = \{x \mid x \in \alpha_0^+ \text{ or } -x \in \alpha_0^+\}$.

$\theta = \{x \mid x \in \theta^+ \text{ or } 1-x \in \theta^+$; or when $2 \in \theta^+$, $x = 2^{-1}\}$.

Proof From the proof of Proposition 3, we know $-1 \in \alpha_0$. From Proposition 1 we know

$$a \in \alpha_0 \Leftrightarrow (-1) \cdot a = -a \in \alpha_0.$$

From the proof of Theorem 2, we know $a \in \theta \Leftrightarrow 1-a \in \theta$. This indicates that the structures of α_0 and θ have a certain "symmetry": From one half, we can get the other. But in set θ, we should think about $1-x=x$ (obviously $x<0$) that is special situation, here $x = 2^{-1}$ and from Proposition 1 we have:

$$2 \in \alpha_0 \Leftrightarrow 2^{-1} \in \alpha_0 \text{ and } 2^{-1}-1 = -2^{-1} \in \alpha_0$$

Notice $1 \in \alpha_0$, there is

$$2 \in \alpha_0 \Leftrightarrow 2 \in \theta^+ \Leftrightarrow 2^{-1} \in \theta.$$

Proof is over.

According to Proposition 4, from α_0^+ we can easily make α_0 and θ^+, and then, we can make θ. With Theorems 2 and 3, we can get a simple, convenient and easily operating mathod. When we study the lower bounds of $R_n(3)$ and $R_n(4)$. Guided by the strict theory, the writer has made a lot of achievments of Theorem 1 with computer.

3 The proof of Theorem 1

Proposition 5 $R_5(3) \geqslant 102$.

Proof Set $n=5$, prime number $p=101$, thus $g=2$ is the minimum primitive root. $g^5 = 32$ is the minimum generator of cyclic groups α_0, we order:

$\alpha_0^+ = \{x \mid x \equiv 2^{5i} \pmod{101}$, and $x>0, 0 \leqslant i < 10\} = \{1, 6, 10, 14, 17, 32, 36, 39, 41, 44\}$

Obviously for any $x \in \alpha_0^+$ there is $x-1 \overline{\in} \alpha_0^+$ for ever, from Theorem 2, we get Proposition 5. Proof is over.

Proposition 6 $R_3(4) \geqslant 128$.

Proof Set $n=3$, prime number $p=127$. Then we get that $g^3=5$ is the minimum generator of cyclic group α_0, thus:

$\alpha_0^+ = \{x \mid x \equiv 5^i \pmod{127}$, and $x>0, 0 \leqslant i < 21\} = \{1, 2, 4, 5, 8, 10, 16, 19, 20, 25, 27, 32, 33, 38, 40, 47, 50, 51, 54, 61, 63\}$

$\theta = \{2, 5, 20, 33, 51, -63, -50, -32, -19, -4, -1\}$

We can easily test and verify: for any $a \in \theta^+ = \{2, 5, 20, 33, 51\}$, $\theta(a) = \varnothing$ for ever. According to Theorem 3, we get Proposition 6. Proof is over.

According to the above, we can prove all results about $R_n(3)$ and $R_n(4)$ in Theorem 1. For simplicity and convenience, we list the n, p and the minimum generator g^n of α_0 and the numbers $|\theta|$ of the elements of set θ about $R_n(4)$ in Theorem 1 as following:

Table 1 About R_n (3)

n	Prime p	g^n
5	101	32
6	277	4
7	491	12
9	577	20
10	1181	4

Table 2 About R_n (4)

n	Prime p	g^n	$\|\theta\|$
3	127	5	11
4	457	6	20
5	941	12	24
6	3457	2	65
8	9697	4	92
10	17681	2	93
12	28297	2	125
14	47797	37	128
20	84961	5	107
22	87869	55	144
24	155377	27	191
28	207481	17	167
29	230551	93	200
30	287701	104	198
32	345089	18	264

All the results of Theorem 1 have been verified and printed out with the computer.

Acknowlegement

Many thanks to Lou Haipeng (recearch fellow, vice-president, Guangxi Academy of Sciences), Huang Suning (vice-director, Guangxi Electronic Teaching Center), Wu Kang (vice Prof. South China Normal University, general secretary of Guangdong Combination Maths Association) for lots of encouragements and advice.

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