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On the Complex Oscillation of Solutions of Second Order Non-homogeneous Linear Differential Equations
关于二阶非齐次线性微分方程解的复振荡

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Abstract Nevanlinna theory is used to investigate the zeros distribution of solutions of $f''+A(z)f'+B(z)f=F(z)$, where $A(z), B(z), F(z)\not\equiv 0$ are all entire functions of finite order of growth, and obtain Theorem 1 and Theorem 2.

Key words second order non-homogeneous linear differential equation, zero-sequence, exponent of convergence

摘要 以 Nevanlinna理论来研究方程 $f''+A(z)f'+B(z)f=F(z)$的解的零点分布，其中 $A(z), B(z), F(z)\not\equiv 0$均为有穷增长级整函数. 得出的主要结果是定理 1和定理 2

关键词 二阶非齐次线性微分方程 零点序列 收敛指数

Using value distribution of Nevanlinna theory to deal with complex oscillation of solutions of linear differential equations is being more active in the world. Especially, to investigate the complex oscillation of non-homogeneous linear differential equations is an important aspect. Consider

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_o(z)f=F(z), \tag{1}$$

where $A_j(z)\,(j=0,1,\cdots,k-1;\ k\geqslant 1)$, $F(z)\not\equiv 0$ are entire functions of finite order. If $A_j(z)$ are all polynomials. then we have obtained many precise results on the complex oscillation of solutions of (1) in references [2, 4, 5, 6, 9]. If $A_j(z)$ are not all polynomials, it is beginning to investigate the complex oscillation of solutions of (1). In this case, we may find some stimulating results in references [3, 7]. Now we only focus our attention on the complex oscillation of solutions of second order non-homogeneous linear differential equations with entire coefficients.

Denote the exponent of covergence of the zero-sequence of entire function $g(z)$ by $\lambda(g)$, the exponent of convergence of the sequence of distinct zeros of $g(z)$ by $\bar{\lambda}(g)$, and the order of growth of $g(z)$ by $e(g)$ in this paper. In an addition, other notations of function theory are standard, e. g. in reference [8].

1 Main results

Consider the equation.

$$f''+A(z)f'+B(z)f=F(z), \tag{2}$$

where $A(z), B(z), F(z)\not\equiv 0$ are all entire functions of finite order of growth. The homogeneous linear differential equation of (2) is as follows.

$$f''+A(z)f'+B(z)f=0. \tag{3}$$

Theorem 1 Let $A(z)$, $B(z)$, $F(z)\not\equiv 0$ be all entire functions of finite order, and $f_1(z), f_2(z)$ be two linearly independent solutions of (3) such that $\lambda(f_1f_2)<\infty$. Then any solution $f(z)$ of (2) must at least satisfy the following

(a) $e(f)<\infty$

(b) $e(f)=\lambda(f)$.

Remark The conditions of Theorem 1 are necessary. For example, consider the equation

$$f''+A(z)f'+(1/4\,A(z)^2+1/2A'(z)+P(z))f=F(z).$$

where $A(z)$ is a transcendental entire function of fi-

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nite order, $P(z)$ is a nonzero polynomial and $F(z) \not\equiv 0$ is an entire function of finite order. Its homogeneous equation is

$$f'' + A(z)f' + (1/4\, A(z)^2 + 1/2A'(z) + P(z))f = 0. \qquad (*)$$

We know any two linearly independent solutions $f_1(z)$, $f_2(z)$ of (*) must satisfy $\lambda(f_1 f_2) < \infty$. In fact, make transformation

$$f = ye^{-1/2\int A(z)dz}$$

and (*) becomes

$$y'' + P(z)y = 0 \qquad (**)$$

It is easy to follow that any two linearly independent solutions $y_1(z), y_2(z)$ of (**) must satisfy $\lambda(y_1 y_2) < \infty$. This means that $\lambda(f_1 f_2) < \infty$.

Theorem 2 Let $A(z)$, $B(z)$, $F(z) \not\equiv 0$ be all entire functions of finite order, and at least one of $A(z)$, $B(z)$ be not a polynomial. And let $f_1(z)$, $f_2(z)$ be two linearly independent solutions of (3), and $f_2 = H(z)\exp\{-\int A(z)dz\}$, where $H(z)$ is a nonzero entire function of finite order. Then any solution $f(z)$ of (2) satisfies

(a) If $\overline{\lambda}(f_1) < \infty$. Then at least one of the following holds:

(i) $e(f) < \infty$;

(ii) $e(f) = \lambda(f)$.

(b) If $\overline{\lambda}(f_1) = \infty$. Then $\lambda(f) = \infty$.

2 Lemmas needed for the Theorems

Lemma 1[8] Let $f(z)$ be a transcendental meromorphic function, and k be a positive integer. Then

$$m(r, f^{(k)}/f) = S(r, f).$$

Lemma 2[1] If $F(r)$ and $G(r)$ are nondecreasing functions on $(0, \infty)$ such that $F(r) \leqslant G(r), r \overline{\in} E$, where E is a set with at most finite measure, then for any constant $T > 1$, there exists $r_0 > 0$ such that $F(r) \leqslant G(Tr)$ for all $r > r_0$.

Lemma 3 Let $f(z)$ be a nontrivial solution of (3) such that $\lambda(f) < \infty$. Then

$$\overline{\lim_{r\to\infty}}[\log^+ m(r, f'/f)/\log r] \leqslant \max\{e(A), e(B)\}.$$

Proof. Since $\lambda(f) < \infty$, we can write

$$f(z) = G(z)e^{g(z)}$$

where $G(z), g(z)$ are both entire functions and $e(G) < \infty$. Substituting $f(z)$ into (3), we get

$$g'^2 = -B - g'(A + g''/g' + 2G'/G) - G''/G - AG'/G.$$

By Nevalinna theory, it follows that

$$m(r, g') \leqslant 2m(r, A) + m(r, B) + S(r, g') + 0\{\log r\},$$

and that,

$$T(r, g') \leqslant 2T(r, A) + m(r, B) + 0\{\log T(r, g')\}, r \overline{\in} E$$

where E is a set of finite measures. By Lemma 2, it is easy to get that

$$e(g') \leqslant \max\{e(A), e(B)\}. \qquad (4)$$

On the other hand, from $f(z)$, it gets

$$f'/f = G'/G + g'.$$

So that,

$$m(r, f'/f) \leqslant T(r, g') + 0\{\log r\}. \qquad (5)$$

From (4) and (5), it is easy to prove Lemma 3.

3 The Proof of Theorem 1

By variation of parameters, for a solution $f(z)$ of (2), we can write

$$f(z) = U_1(z)f_1(z) + U_2(z)f_2(z), \qquad (6)$$

where $U_1(z), U_2(z)$ are determined by

$$U_1'f_1 + U_2'f_2 \equiv 0, \qquad (7)$$

$$U_1'f_1 + U_2'f_2' \equiv F. \qquad (8)$$

From (7), (8) and Wronskian of f_1 and f_2:

$$W(f_1, f_2) = e^{-\int A(z)dz},$$

it gives

$$U_1' = -f_2Fe^{\int A(z)dz}. \qquad (9)$$

From (6) it follows

$$(f - U_1f_1)f_2^{-1} = U_2.$$

Differentiating two sides of the above, we obtain

$$(f' - U_1'f_1 - U_1f_1')f_2^{-1} - (f - U_1f_1)f_2'f_2^{-2} = U_2'.$$

Combining (7), we have

$$U_1 = (ff_2'/f_2 - f')(f_1f_2'/f_2 - f_1')^{-1}.$$

Differentiating the above, we get

$$U_1'f_1(f_2'/f_2 - f_1'/f_1) = f(f_2'/f_2 - f'/f)\left\{\frac{(ff_2'/f_2 - f')'}{ff_2'/f_2 - f'} - \frac{(f_1f_2'/f_2 - f_1')'}{f_1f_2'/f_2 - f_1'}\right\}$$

From (9), it follows that

$$-f_1f_2(f_2'/f_2 - f_1'/f_1)Fe^{\int A(z)dz} = f(f_2'/f_2 - f'/f)\left\{\frac{(ff_2'/f_2 - f')'}{ff_2'/f_2 - f'} - \frac{(f_1f_2'/f_2 - f_1')'}{f_1f_2'/f_2 - f_1'}\right\}. \qquad (10)$$

On the contrary, by Wronskian of f_1 and f_2, from (3), it follows

$$f_1f_2(f_2'/f_2 - f_1'/f_1) = e^{-\int A(z)dz}.$$

Substituting the above into (10), we have

$$-F = f(f_2'/f_2 - f'/f)\left\{\frac{(ff_2'/f_2 - f')'}{ff_2'/f_2 - f'} - \frac{(f_1f_2'/f_2 - f_1')'}{f_1f_2'/f_2 - f_1'}\right\}$$

And that

$$f^{-1} = F^{-1}(f'/f - f_2'/f_2)\left\{\frac{(ff_2'/f_2 - f')'}{ff_2'/f_2 - f'} - \frac{(f_1f_2'/f_2 - f_1')'}{f_1f_2'/f_2 - f_1'}\right\}$$

Applying Nevanlinna theory to the above, it gets

$$m(r,f^{-1}) \leqslant m(r,F^{-1}) + S_0(r),$$

and then

$$T(r,f) \leqslant N(r,1/f) + T(r,F) + S_0(r),$$

where $S_0(r) = S(r,f_1) + S(r,f_2) + S(r,f)$. By Lemma 1, it gives

$$\sum_{j=1}^{2} S(r,f_j) = \sum_{j=1}^{2} m(r,f_j'/f_j)$$

So that

$$T(r,f) \leqslant N(r,1/f) + T(r,F) + \sum_{j=1}^{2} m(r,f_j'/f_j) + O\{\log rT(r,f)\} \quad r \overline{\in} E,$$

where E is a set of finite measures. By Lemma 2 and Lemma 3, it easily follows that

$$\mathrm{e}(f) \leqslant \max\{\lambda(f), \mathrm{e}(F), \mathrm{e}(A), \mathrm{e}(B)\}.$$

If $\lambda(f) < \max\{\mathrm{e}(F), \mathrm{e}(A), \mathrm{e}(B)\}$. Then $\mathrm{e}(f) < \infty$.

If $\lambda(f) \geqslant \max\{\mathrm{e}(F), \mathrm{e}(A), \mathrm{e}(B)\}$, noting that $\lambda(f) \leqslant \mathrm{e}(f)$. Then $\mathrm{e}(f) = \lambda(f)$.

Now, Theorem 1 is completely proved.

4 The Proof of Theorem 2

(a) If $\overline{\lambda}(f_1) < \infty$. Then we assert that $\lambda(f_1) < \infty$, and by Theorem 1 the conclusion is true. In fact, if assume $\lambda(f_1) = \infty$. Then $\mathrm{e}(f_2) = \infty$, We shall point out this case is impossible. Similarly, from the proof of Theorem 1, it is easy to deduce the following:

$$f(z) = \mathrm{T}_1(z)f_1(z) + \mathrm{T}_2(z)f_2(z), \tag{11}$$

$$\mathrm{T}_1'f_1 + \mathrm{T}_2'f_2 \equiv 0, \tag{12}$$

$$\mathrm{T}_1'f_1' + \mathrm{T}_2'f_2' \equiv F, \tag{13}$$

$$\mathrm{T}_1' = -F_2 f e^{\int A(z)dz}, \tag{14}$$

$$\mathrm{T}_2' = f_2 F e^{\int A(z)dz}. \tag{15}$$

Substituting $f_2(z)$ into (14), we have

$$\mathrm{e}(\mathrm{T}_1') < \infty, \tag{16}$$

and

$$\mathrm{e}(\mathrm{T}_1) < \infty. \tag{17}$$

Since

$$f_2' = (H' - HA)e^{-\int A(z)dz}, \tag{18}$$

where $H' - HA \not\equiv 0$. Otherwise, assume that $H' - HA \equiv 0$, this implies that $A(z)$ is a polynomial and f_2 is a constant, from (3), it deduces that B(z) is a constant. This contradicts the hypothesis. From (13), (15) and (18), it gives

$$\mathrm{T}_1'f_1' + f_1F(H' - HA) = F.$$

So

$$f_1 = F(FH' - FHA + \mathrm{T}_1'f_1'/f_1)^{-1}.$$

By Nevanlinna theory, it follows

$$T(r,f_1) \leqslant N(r,1/f) + 0\{T(r,F) + T(r,H) + T(r,\mathrm{T}_1') + \log rT(r,f_1)\} \quad r \overline{\in} E,$$

where E is a set of finite measures. Noting that $\lambda(f_1) < \infty$, $\mathrm{T}(F) < \infty$, $\mathrm{T}(H) < \infty$ and (16), by Lemma 2, it easily has that

$$\mathrm{e}(f_1) < \infty.$$

This is a contradiction to our assumption. This means that our assertion is reasonable.

(b) From (11), that

$$(f - \mathrm{T}_1 f_1)f_2^{-1} = \mathrm{T}_2.$$

Differentiating the above, we obtain

$$(f' - \mathrm{T}_1'f_1 - \mathrm{T}_1 f_1')f_2^{-1} - (f - \mathrm{T}_1 f_1)f_2'f_2^{-2} = \mathrm{T}_2'.$$

Combinig (15), we have

$$(f' - \mathrm{T}_1'f_1 - \mathrm{T}_1 f_1')f_2^{-1} - (f - \mathrm{T}_1 f_1)f_2'f_2^{-2} = f_1 F e^{\int A(z)dz}$$

Substituting $f_2(z)$ into the above, we get

$$f' - \mathrm{T}_1'f_1 - \mathrm{T}_1 f_1' - (f - \mathrm{T}_1 f_1)(H'/H - A) = f_1 FH.$$

And that

$$f = [f_1(FH + \mathrm{T}_1' + \mathrm{T}_1 A - \mathrm{T}_1 H'/H) + \mathrm{T}_1 f_1'](f'/f + A - H'/H)^{-1}. \tag{19}$$

From (13), (15) and (18), it has

$$f_1' = [F - (H' - HA)f_1F]/\mathrm{T}_1'.$$

Substituting f_1' into (19), and noting that $\mathrm{T}_1' = -HF$, we obtain

$$f_1 = \frac{f(f'/f + A - H'/H) - \mathrm{T}_1 F/\mathrm{T}_1'}{\mathrm{T}_1 A - \mathrm{T}_1 H'/H - (H' - HA)F\mathrm{T}_1/\mathrm{T}_1'}.$$

By Nevanlinna theory, it easily deduces that

$$T(r,f_1) \leqslant 0\{T(r,f) + T(r,H) + T(r,A) + T(r,\mathrm{T}_1) + T(r,F)\} \quad r \overline{\in} E,$$

where E is a set of finite measures. Since $\lambda(f_1) = \infty$, $\mathrm{e}(f_1) = \infty$. By Lemma 2, and noting that $\mathrm{e}(H) < \infty$, $\mathrm{e}(A) < \infty$, $\mathrm{e}(F) < \infty$ and (17), we conclude that

$$\mathrm{e}(f) = \infty.$$

From (2), it gets

$$f''/f + A(z)f'/f + B(z) = F/f.$$

By Nevanlinna theory, it has

$$m(r,F/f) \leqslant m(r,A) + m(r,B) + S(r,f),$$

and then

$$T(r,F/f) \leqslant N(r,1/f) + T(r,A) + T(r,B) + S(r,f).$$

And that,

$$T(r,f) \leqslant N(r,1/f) + T(r,F) + T(r,A) + T(r,B) + S(r,f).$$

So, it easily follows that

$$\lambda(f) = \infty$$

Now, Theorem 2 is completely proved.

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3) $m \cdot \max_{1 \leqslant j \leqslant m}\{\tau_j\} \cdot C < a$

4) 当 $|x| \to \infty$ 时, $[f(x) + \sum_{j=1}^{m} g_j(x)]\operatorname{sgn} x \to \infty$

则 1) 系统(3)的运动是一致最终有界的.

2) 若还有 $H(t+T, y) = H(t, y)$ 对任意 $(t, y) \in R^2$ 成立,且 $p(t)$ 是周期为 T 的周期函数,那么系统(3)存在周期为 T 的周期运动.

3) 若系统(3)的任意两个运动 $x_1(t)$ 与 $x_2(t)$ 均满足 $x_1(t) - x_2(t) \to 0 (t \to \infty)$,那么系统(3)存在平稳振荡.

证明 注意到系统(3)等价于

$$\begin{cases} \dot{x} = y \\ \dot{y} = -[f(x) + \sum_{j=1}^{m} g_j(x)] - H(t, y) \\ \qquad + \sum_{j=1}^{m} \int_{-\tau_j}^{0} g_j'(x(t+s)) y(t+s) \mathrm{d}s + p(t) \end{cases}$$

由定理4、5、6即得本定理结论.

2.2 考虑具有时滞恢复力 $\sum_{j=1}^{m} [\alpha_j \sin x(t-\tau_j) + \beta_j x(t-\tau_j)]$,外力是 $p(t)$ 和摩擦与速度成比例的反馈系统

$$\ddot{x}(t) + a\dot{x}(t)) + \sum_{j=1}^{m} [\alpha_j \sin x(t-\tau_j) + \beta_j x(t-\tau_j)] = p(t) \quad (5)$$

这里 $a > 0$, $\beta_j > 0 (j = 1, 2, \cdots, m)$, $\alpha_j \in R (j = 1, 2, \cdots, m)$, $\tau_j > 0$ 为常数.

我们将定理7应用于系统(5),其中 $H(t, \dot{x}(t)) = a\dot{x}(t)$, $f(x(t)) \equiv 0$, $g_j(x(t-\tau_j)) = \alpha_j \sin(x-\tau_j) + \beta_j(x-\tau_j)$ $(j = 1, 2, \cdots, m)$.

定理8 Ⅰ 当 $p(t) \equiv 0$(外力为零)时,若

1) $|\sum_{j=1}^{m} \alpha_j| + \sum_{j=1}^{m} \beta_j > 0$

2) $m \cdot \max_{1 \leqslant j \leqslant m}\{|\alpha_j| + |\beta_j|\} \cdot \max_{1 \leqslant j \leqslant m}\{\tau_j\} < a \quad (6)$

则系统(5)的运动稳定、一致稳定、渐近稳定和一致渐近稳定.

Ⅱ 若 $p(t)$ 连续有界且不等式(6)成立,则

1) 系统(5)的运动是一致最终有界的.

2) 此外,若 $p(t+T) = p(t)$,那么系统(5)存在周期为 T 的周期运动.

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