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A Stage-structured Single Species Model with Diffusion*
一类具有阶段结构的单种群扩散模型

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Abstract　A stage-structured single species model with diffusion is considered, where all the parameters are constants. The system, which is composed of two patches, has one species which is divided into immature and mature. And the mature can diffuse between the two patches while the immature population is confined to one patch and cannot diffuse. We get the boundary of the solution and the existence of the positive equilibrium of the model, and obtain the asymptotically stabilities of the positive equilibrium of the model and the positive equilibrium under proper conditions. we prove that the unstability of the spot (0, 0, 0).

Key words　stage-structured, diffusion, positive equilibrium, asymptotically stable

摘要　提出并研究了一类具有阶段结构的单种群扩散模型,得到其在正平衡点处的稳定性和在零平衡点处的不稳定性.

关键词　阶段结构　扩散　正平衡　渐近稳定性
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The study about single species density dependent models was early found in 1938 when Verhulst-pearl established the following equation[1]

$$\frac{dN(t)}{dt} = \frac{r_m}{K} N(t)(K - N(t))$$

Here, $N(t)$ is the density of the species in time t; r_m is the intrinsic rate of natural increase and K is the carrying capacity of the system. It was proved in the work that the positive equilibrium was unique and global stable.

Though the model above approaches to the real more close than the density independent model, its faults are apparent. First, it is unfit to the species which can diffuse among several patches; second, the life stage structure of the species is ignored in such a model.

Later, in 1986, Freedman established the diffusive single species model[2], but the stage structure was still ignored. Since for many species, it is well known to all that the rate of death and the ability of diffusion and predator is greatly different, thus it is significant to study the stage-structured model. In recent years, such work appeared in reference [3~ 5], however, in their models, the species are confined in one close environment and cannot diffuse. The stage-structured single species model with diffusion, which is more realistic than all of the above, still remains unstudied. Therefore it is significant to propose and consider such model in our paper.

$$\begin{cases} \dot{x}_1 = bx_2 - gx_1 - dx_1^2; \\ \dot{x}_2 = gx_1 - D_1x_2^2 + \lambda_1(y_2 - x_2); \\ \dot{y}_2 = - D_2y_2^2 + \lambda_2(x_2 - y_2). \end{cases} \tag{1}$$

Here, we classity the single species into immature and mature. $x_1(t)$ is the density of the immature and $x_2(t)$ is that of mature in patch 1, $y_2(t)$ is the density of the mature in patch 2. In our model, b is the birth rate into immature and g is the growth rate into mature from immature. We assume the mature can diffuse between patch 1 and 2, and λ_1, λ_2 are the diffusion coefficients. For the immature, it is confined in patch 1 and cannot diffuse. D_1, D_2 are the density dependent coefficients of the mature and d is that of the immature.

For the ecological meaning, we assume the system satisfies:

$$b > 0, g > 0, d > 0, D_1 > 0, D_2 > 0, \lambda_1 > 0, \lambda_2 > 0 \tag{H1}$$

1 Main results

Lemma 1　Suppose (H1) holds true, then $R_+^3 = \{(x_1, x_2, y_2) | x_1 > 0, x_2 > 0, y_2 > 0\}$ is a positive invariant set of (1).

Proof　The system (1) will be discussed by the following three steps.

(i) $\dot{x}_1|_{x_1=0, x_2>0} = bx_2 > 0$.

(ii) $\dot{x}_2|_{x_2=0, x_1>0, y_2>0} = gx_1 + \lambda_1 y_2 > 0$.

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(iii) $\dot{y}_2|_{x_2>0, y_2=0} = \lambda_2 x_2 > 0$.

From (i), (ii) and (iii), the assertion of the lemma follows immediately for all $t \in [0, +\infty]$. That completes the proof.

Theorem 1 Assume system (1) satisfies (H_1) and the condition $b - g > 0$, then there exists a compact region $K \subset R_+^3$ such that for each solution $(x_1(t), x_2(t), y_2(t))$ with positive initial value $x_i(0) > 0$, $y_2(0) > 0$, $(i = 1, 2)$. There exists $T > 0$ such that $\{x_1(t), x_2(t), y_2(t)\} \in K$ for all $t \geq T$.

Proof Let

$$f(t) = \max\{x_1(t), x_2(t), y_2(t)\},$$

$$M^* = \max\{\frac{b-g}{d}, \frac{g}{D_1}\}.$$

Obviously $f(0) = \max\{x_1(0), x_2(0), y_2(0)\}$. By Lemma 1 $f(t) > 0$ and there exist some $t \geq 0$ and a function $k(t)$ in $x_1(t)$, $x_2(t)$ and $y_2(t)$ such that $f(t) = k(t)$ for the t. It is easy to see that $f(t)$ is continuous and right upper derivable. Now, we caculate and estimate the right upper derivative of $f(t)$ as the following two cases:

Case (i) If $f(0) \leqslant M$, here M is positive constant and satisfies $M > M^*$. We consider as follows.

(a) When $f(t) = x_1(t) (t \geq 0)$, it comes from the first equation of system (1) that

$$D^+ f(t)|_{f(t)=M} \leqslant bM - gM - dM^2 = dM(\frac{b-g}{d} - M) \leqslant dM(M^* - M) < 0.$$

(b) When $f(t) = x_2(t)$, it comes from the second equation of system (1) that

$$D^+ f(t)|_{f(t)=M} \leqslant gM - D_1 M^2 = D_1 M(\frac{g}{D_1} - M) \leqslant D_1 M(M^* - M) < 0.$$

(c) When $f(t) = y_2(t)$, from the last equation of system (1) it comes that

$$D^+ f(t)|_{f(t)=M} \leqslant -D_2 M^2 < 0.$$

Let

$$\mathrm{T} = -\min\{dM(M^* - M), MD_1(M^* - M), -D_2 M^2\}.$$

Then $\mathrm{T} > 0$ and by (a), (b), (c), $f(0) \leqslant M \Rightarrow f(t) \leqslant M$ for all $t \geq 0$. Case (i) is completed.

Case (ii) If $f(0) > M$, by the similar proof to Case (i), we can have

$$D^+ f(t)|_{f(t)>M} < -\mathrm{T} < 0.$$

Then $f(t)$ will monotonously decrease by the speed more than T if $f(0) > M$. Let $T_1 = \frac{f(0) - M}{\mathrm{T}}$, now we prove there must exist a positive constant T in $(0, T_1]$ such that $f(T) = M$.

If, for all $t \in [0, T_1]$, $f(t) > M$ holds, then we have $D^+ f(t) < -\mathrm{T} < 0 (t \in [0, T_1])$. An integration of the inequation about t on $[0, T_1]$ leads to $f(T_1) - f(0) < -\mathrm{T} T_1 = M - f(0)$, then $f(T_1) < M$. A contradiction to our assumption. This prove that there exists a constant T which satisfies $0 < T \leqslant T_1$ and $f(T) = M$. Therefore by Case (i) we obtain $f(t) \leqslant M$ for all $t \geq T$. Case (ii) is compeleted.

Let

$$K = \{\{x_1(t), x_2(t), y_2(t)\} | 0 \leqslant x_1(t), x_2(t), y_2(t) \leqslant M\}$$

By Case (i) and (ii), for all the solution of (1) with positive initial values, we have

$$\{x_1(t), x_2(t), y_2(t)\} \in K (t \geq T)$$

That completes the proof.

The following Lemma is about the existence of the positive equilibrium of system (1), it is

Lemma 2 There is at least one positive equilibrium of system (1) which is denoted as $E = (x_1^*, x_2^*, y_2^*)$, $(x_1^* > 0, x_2^* > 0, y_2^* > 0)$.

Proof Let

$$\begin{cases} bx_2 - gx_1 - dx_1^2 = 0; \\ gx_1 - D_1 x_2^2 + \lambda_1 (y_2 - x_2) = 0; \\ -D_2 y_2^2 + \lambda_2 (x_2 - y_2) = 0. \end{cases} \quad (2)$$

From the second and the third equations of system (2) we get

$$\frac{g}{\lambda_1} x_1 = \frac{D_1}{\lambda_1} x_2^2 + \frac{D_2}{\lambda_2} y_2^2. \quad (3)$$

While by the third equation we obtain

$$x_2 = \frac{D_2}{\lambda_2} y_2^2 + y_2. \quad (4)$$

Substitute (3), (4) into the first equation in (2) we have

$$\frac{d\lambda_1^2}{g^2} y_2^3 [\frac{D_2}{\lambda^2} + \frac{D_1}{\lambda^1} (\frac{D_2}{\lambda_2} y_2 + 1)^2]^2 + \lambda_1 y_2 [\frac{D_1}{\lambda_1} (\frac{D_2}{\lambda^2} y_2 + 1)^2 + \frac{D_2}{\lambda^2}] - b(\frac{D^2}{\lambda^2} y_2 + 1) = 0. \quad (5)$$

Clearly, the left part of (5) is a seventh-order multinomial where the coefficient of its first nomial is positive. Let $f(y_2)$ be the left part of the equation (5), we obtain

$$f(0) = -b < 0, f(y_2) \to +\infty \text{ when } y_2 \to +\infty.$$

Then there exists at least one $y_2^* > 0$ such that $f(y_2^*) = 0$. By (3), (4) we know $x_1^* > 0$, $x_2^* > 0$. Denote $E = (x_1^*, x_2^*, y_2^*)$, then the proof is completed.

Now, by making use of Lemma 2, it comes the following theorem.

Theorem 2 Assume (H_1) and

$$\lambda_1 + \lambda_2 > b \quad (H_2)$$

hold, then every positive equilibrium of system (1) is asymptotically stable.

Proof By Lemma 2, the existence of positive equilibrium of (1) is assured. From system (1) the Jacobian matrix on E is presented as follows.

$$J(E) = \begin{pmatrix} -g - 2dx_1^* & b & 0 \\ g & -\lambda_1 - 2D_1 x_2^* & \lambda_1 \\ 0 & \lambda_2 & -\lambda_2 - 2D_2 y_2^* \end{pmatrix}.$$

Then, the characteristic multinomial of $J(E)$ is as follows.

$$|\lambda I - J(E)| = (\lambda + g + 2d_1 x_1^*)[(\lambda + \lambda_1 + 2D_1 x_2^*)(\lambda + \lambda_2 + 2D_2 y_2^*) - \lambda_1 \lambda_2] - gb(\lambda + \lambda_2 + 2D_2 y_2^*).$$

Let $A = g + 2d_1x_1^*$, $B = \lambda_1 + 2D_1x_2^*$, $C = \lambda_2 + 2D_2y_2^*$, then $A,B,C > 0$ and

$$|\lambda I - J(E)| = \lambda^3 + (A + B + C)\lambda^2 - \lambda^1\lambda^2 A - gbC + [AB + BC + AC - (\lambda_1\lambda_2 + gb)]\lambda + ABC$$

Noticing $BC > \lambda_1\lambda_2$, then it leads to

$$\begin{aligned}&(A + B + C)[AB + BC + AC - (\lambda_1\lambda_2 + gb)] - (ABC - \lambda_1\lambda_2 A - gbC)\\ &= 2ABC + \lambda_1\lambda_2 A + gbC - \lambda_1\lambda_2(A + B + C) - gb(A + B + C) + A^2(B + C) + B^2(A + C) + C^2(A + B)\\ &= 2ABC + BC(BC - \lambda_1\lambda_2) + C(BC - \lambda_1\lambda_2) + A^2(B + C) + B^2A + AC^2 - gb(A + B)\\ &> ABC + (ABC + A^2B + A^2C + A^2B) - gb(A + B) + AC^2 > (A + B)[A(B + C) - gb] > (A + B)(g\lambda_1 + g\lambda_2 - gb) = g(A + B)(\lambda_1 + \lambda_2 - b) > 0.\end{aligned}$$

Therefore, by Routh-Huruitz Theorem, the characteristic roots of $J(E)$ have negative real parts, then there exists a constant $c > 0$ such that all the real parts of characteristic roots are smaller than $-c$. Thus, from reference [6], Theorem 1, system (1) is asymptotically stable on E. Such competes the proof.

We can also get the following Theorem:

Theorem 3 Assume (H_1) holds, then the point (0, 0, 0) is the unstable equilibrium of system (1).

Proof By system (1), the Jacobian matrix on (0, 0, 0) is

$$J(0) = \begin{pmatrix} -g & b & 0 \\ g & -\lambda_1 & \lambda_1 \\ 0 & \lambda_2 & -\lambda_2 \end{pmatrix}.$$

Then its characteristic multinomial of $J(0)$ is

$$F(\lambda) = |\lambda I - J(0)| = (\lambda + g)[(\lambda + \lambda_1)(\lambda + \lambda_2) - \lambda_1\lambda_2] - bg(\lambda + \lambda_2) = \lambda^3 + (g + \lambda_1 + \lambda_2)\lambda^2 + [g(\lambda_1 + \lambda_2) - gb]\lambda - gb\lambda_2.$$

Since $F(0) = -gb\lambda_2 < 0$, $F(+\infty) = +\infty$, there must exist a constant $\lambda^* > 0$ such that $F(\lambda^*) = 0$. Therefore from reference [6], (0, 0, 0) is the unstable equilibrium of the system. Thus completes the proof.

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一类变系数 KdV 型方程的孤立波解

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下列变系数 KdV 型方程描述了一个水深变化的浅水河道中的突变现象[1]

$$u_t + T(t)uu_x + U(t)u_{xxx} = 0, \quad (1)$$

其中 $T(t)$ 和 $U(t)$ 都为可微函数.在本文中,我们构造(1)的如下形式的解

$$u(t,x) = B(t) + A(t)F(s) - A(t)F(s)^2, \quad (2)$$

其中 $F(s) = 1/(1 + \exp s)$, $s(t,x)$, $B(t)$ 和 $A(t)$ 为待定函数,并且 $s_{xx} = 0$.把(2)代入(1),比较 F^j 的系数,我们可以得到:

$$F^5: 2A^2s_xT - 24As_x^3U = 0; \quad (3)$$

$$F^4: 60As_x^3U - 5A^2s_xT = 0; \quad (4)$$

$$F^3: -50As_x^3U + (4A^2 - 2BA)s_xT - 2As_t = 0; \quad (5)$$

$$F^2: 15As_x^3U - (A^2 - 3BA)s_xT + 3As_t - A_t = 0; \quad (6)$$

$$F^1: As_x^3U + BAs_xT + As_t - A_t = 0; \quad (7)$$

$$F^0: B_t = 0. \quad (8)$$

由式(3)~(8)我们得到 B 为一个任意常数,并且

$$A = 12\frac{U(t)}{T(t)}s_x^2,\ A_t = 0,\ s_t = -Us_x^3 - BTs_x, \quad (9)$$

由此得到 $U(t) = lT(t)$,其中 l 为一个任意常数.由(9)和 $s_{xx} = 0$,我们可以得到

$$s(t,x) = kx - (k^3l + kB)\int T(t)\,dt, \quad (10)$$

其中 k 为一个非零常数.容易证明 $F(s) - F(s)^2 = (1/4)\operatorname{sech}^2(s/2)$.因而,如果满足条件 $U(t) = lT(t)$,则方程(1)有如下孤立波解

$$u(t,x) = B + 3lk^2\operatorname{sec}h^2(\frac{1}{2}s), \quad (11)$$

其中 s 如(10)所示.文献[1]中的结果不包含解(11).

我们发现,这种直接方法还可以应用于许多变系数的反应扩散方程的求解.

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