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一类无限时滞微分方程的一致渐近稳定性*

Uniformly Asymptotic Stability for Certain Differential Equations with Infinite Delay

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摘要 考虑如下无限时滞微分方程: $x'(t)+\lambda x(t)=F(t,x_t)$,其中 $\lambda>0$, F: $[0,\infty)\times BC(H)\rightarrow R$ 连续,获得了零解一致稳定与一致渐近稳定的充分条件,推广了文献 [5] 的结果.

关键词 无限时滞 一致稳定性 一致渐近稳定性

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Abstract Consider the following equation $x'(t)+\lambda x(t)=F(t,x_t)$, where $\lambda>0$, F: $[0,\infty)\times BC(H)\rightarrow R$ continuous, we obtain sufficient conditions for the uniform stability and uniformly asymptotic stability of the zero solution of the equation, the results in paper [5] are extended.

Key words infinite delay, uniform stability, uniformly asymptotic stability

考虑时滞微分方程

$$x'(t)+\lambda x(t)=F(t,x_t), \tag{1}$$

其中 F: $[0,\infty)\times BC(H)\rightarrow R$ 连续, $F(\cdot,0)\equiv 0$, $BC(H)=\{h$: $(-\infty,0]\rightarrow R$ 连续且 $\|h\|=\sup_{s\leqslant 0}|h(s)|\leqslant H\}$,对 $h\in BC(H)$ 及 $u\geqslant 0$,定义泛函

$$M_u(h)=\max\{0,\max_{s\in[-u,0]}h(s)\},$$

$$N_u(h)=\max_{s\in[-u,0]}h(s).$$

设 H 表示由有界非减,左连续且恒不为常数的函数 $\_$: $[0,\infty)\rightarrow[0,\infty)$ 组成的函数空间,对于 $\_\in$ H,定义

$$\_{}_0=\int_0^\infty d\_(u),\quad \_{}_1=\int_0^\infty e^{\lambda u}d\_(u).$$

本文总假定对所有 $h\in BC(H)$, F 满足

$$-\int_0^\infty N_u(h)d\_(u)\leqslant F(t,h)\leqslant\int_0^\infty N_u(-h)d\_(u), \tag{2}$$

或者

$$-\int_0^\infty M_u(h)d\_(u)\leqslant F(t,h)\leqslant\int_0^\infty M_u(-h)d\_(u), \tag{3}$$

此外,还假定 F 满足一定条件,以保证 (1) 的解 $x(t;t_0,h)$ 在 $[t_0,+\infty)$ 上存在且唯一 (参见文献 [1]).

当 F 具有有限时滞且满足 York 条件 [2,3],文献 [3,4] 分别获得了 $\lambda=0$ 及 $\lambda\geqslant 0$ 时,(1) 的零解一致渐近稳定的结果,但该方法不能处理无限时滞方程,且当 F 含有多个时滞时,不能反映各时滞在稳定性中所起的作用.针对这个问题,文献 [5] 对 F 作了 (2), (3) 两个假设,使之能处理无限时滞及多个时滞的情况,其主要结果如下:若 $\lambda=0$ 及 (2) 成立,当

$$\int_0^\infty u d\_(u)\leqslant(<)\frac{3}{2}, \tag{4}$$

(1) 的零解一致稳定 (一致渐近稳定).

若 $\lambda=0$ 及 (3) 成立,当

$$\int_0^\infty u d\_(u)\leqslant(<)1, \tag{5}$$

(1) 的零解一致稳定 (一致渐近稳定性,保证一致渐近稳定性,还需对附加收敛条件,参见文献 [5]).

本文将考虑 $\lambda>0$ 的情形,为方便起见,记

$$\_{}_2=\frac{\_{}_0}{\lambda^2}f\left(\frac{\lambda}{\_{}_0}\right)\int_0^{-\frac{1}{\lambda}\ln\left(1-\frac{\lambda}{\_{}_0}\right)}e^{\lambda u}d\_(u),$$

$$\_{}_3=\frac{\_{}_0}{\lambda^2}g\left(\frac{\lambda}{\_{}_0}\right)\int_0^{-\frac{1}{\lambda}\ln\left(1-\frac{\lambda}{\_{}_0}\right)}d\_(u),$$

其中 $f(x)=x-\frac{1}{2}x^2+(1-x)\ln(1-x)$, $g(x)=x+(1-x)\ln(1-x)$.

主要结果如下:

定理 1 设 (2) 式成立,

(i) 当

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$$\lambda \geqslant \_{0} \text{或} \lambda < \_{0} \text{且} \frac{1}{\lambda}(\_{1} - \_{0}) \leqslant 1 + \frac{1}{2\_{0}} + \_{2}, \tag{6}$$

(1) 的零解一致稳定;

(ii) 当

$$\lambda \geqslant \_{0} \text{或} \lambda < \_{0} \text{且} \frac{1}{\lambda}(\_{1} - \_{0}) < 1 + \frac{1}{2\_{0}} + \_{2}, \tag{7}$$

(1) 的零解一致渐近稳定.

定理 **2** 设 (3) 成立,

(i) 当

$$\lambda \geqslant \_{0} \text{或} \lambda < \_{0} \text{且} \frac{1}{\lambda}(\_{1} - \_{0}) + \_{3} \leqslant 1 + \frac{1}{2\_{0}} + \_{2}, \tag{8}$$

(1) 的零解一致稳定;

(ii) 当

$$\lambda \geqslant \_{0} \text{或} \lambda < \_{0} \text{且} \frac{1}{\lambda}(\_{1} - \_{0}) + \_{3} < 1 + \frac{1}{2\_{0}} + \_{2}. \tag{9}$$

推论 **1** 设 (2) 成立,

(i) 当 $\lambda \geqslant \_{0}$,

(1) 的零解一致渐近稳定;

(ii) 当 $\lambda < \_{0}$ 且

$$\frac{1}{\lambda}(\_{1} - \_{0}) \leqslant (<) 1 + \frac{1}{2\_{0}}, \tag{10}$$

(1) 的零解一致稳定 (一致渐近稳定).

推论 **2** 设 (3) 成立,

(i) 当 $\lambda \geqslant \_{0}$,

(1) 的零解一致渐近稳定;

(ii) 当 $\lambda < \_{0}$ 且

$$\frac{1}{\lambda}(\_{1} - \_{0}) + \frac{\_{0}^{2}}{\lambda^{2}} g\left(\frac{\lambda}{\_{0}}\right) \leqslant (<) 1 + \frac{1}{2\_{0}}, \tag{11}$$

(1) 的零解一致稳定 (一致渐近稳定).

在推论 1、推论 2中,若将 $\_{0}$ 固定,$\lambda \to 0$,则 $\frac{1}{\lambda}(\_{1} - \_{0}) \to \int_{0}^{\infty} u \, d\_(u)$,$\frac{1}{2\_{0}} \to \frac{1}{2}$,$\frac{\_{0}^{2}}{\lambda^{2}} g(\lambda\_{0}) \to \frac{1}{2}$. 此时,条件 (10)、(11) 分别变为条件 (4)、(5). 从这个意义上讲,我们的结果是文献 [5]的推广.

1 定理的证明

引理 **1** 设 (2)、(6) 成立或者 (3)、(8) 成立,则 $t \geqslant t_0 + \frac{1}{\lambda} \ln\left(1 + \frac{\lambda}{\_{0}}\right)$, $0 < |x(t)| = \| x_t \|$, $\frac{d}{dt}|x(t)| > 0$ 蕴含着 $|x(s)| > 0$, $s \in \left[t_0 + \frac{1}{\lambda} \ln\left(1 + \frac{\lambda}{\_{0}}\right), t\right]$.

证明 不妨设 $x(t) > 0$,假设上述论断不成立,则存在 $t_1 \in \left[t_0 + \frac{1}{\lambda} \ln\left(1 + \frac{\lambda}{\_{0}}\right), t\right)$ 使得 $x(t_1) = 0$, $x(s) > 0$, $s \in (t_1, t]$. 由 $x'(t) > 0$, 可选取 t^*, 使得 $x(s) > x(t)$, $s \in (t, t^*]$, 下记 $X = \| x_t \|$.

由 (1)、(2) 或 (3)

$$|(e^{\lambda s} x(s))'| = |e^{\lambda s} F(s, x_s)| \leqslant \_{0} e^{\lambda s} X, s \in [t_0, t], \tag{12}$$

于是

$$e^{\lambda t} X = e^{\lambda t} x(t) = \int_{t_1}^{t} (e^{\lambda s} x(s))' ds \leqslant \frac{\_{0}}{\lambda}(e^{\lambda t} - e^{\lambda t_1}) X,$$

从而

$$\frac{\lambda}{\_{0}} \leqslant 1 - e^{-\lambda(t - t_1)}, \tag{13}$$

由此可知 $\lambda < \_{0}$,否则将导出矛盾,以下设 $\lambda < \_{0}$,取 t_2 使得

$$1 - e^{-\lambda(t^* - t_2)} = \frac{\lambda}{\_{0}}, \tag{14}$$

由 (17) 及 $t^* > t$,可知 $t_2 > t_1$,

当 $s \in [t_0, t_1]$ 时,

$$-e^{\lambda s} x(s) = \int_{s}^{t_1} (e^{\lambda f} x(f))' df \leqslant \int_{s}^{t_1} e^{\lambda f} df \cdot \_{0} X$$

$$= \frac{\_{0}}{\lambda}(e^{\lambda t_1} - e^{\lambda s}) X,$$

故 $x(s) \geqslant -\frac{\_{0}}{\lambda}(e^{\lambda(t_1 - s)} - 1) X$.

当 $s \in [t_1, t^*]$ 时,

$$e^{\lambda t} X - e^{\lambda s} x(s) = \int_{s}^{t} (e^{\lambda f} x(f))' df \leqslant \int_{s}^{t} \_{0} e^{\lambda f} df X$$

$$= \frac{\_{0}}{\lambda}(e^{\lambda t} - e^{\lambda s}) X.$$

故 $x(s) \geqslant X\left(\frac{\_{0}}{\lambda} - \left(\frac{\_{0}}{\lambda} - 1\right) e^{\lambda(t - s)}\right)$.

由于 $x(s) > X$, $s \in (t, t^*]$, 从而

$$x(s) \geqslant X\left(\frac{\_{0}}{\lambda} - \left(\frac{\_{0}}{\lambda} - 1\right) e^{\lambda(t^* - s)}\right), s \in [t_1, t^*].$$

定义

$$d(s) = \begin{cases} X\left[\frac{\_{0}}{\lambda} - \left(\frac{\_{0}}{\lambda} - 1\right) e^{\lambda(t^* - s)}\right], & s \in [t_2, t^*] \\ 0, & s \in [t_1, t_2] \\ -X \min\left\{1, \frac{\_{0}}{\lambda}(e^{\lambda(t_1 - s)} - 1)\right\}, & s \leqslant t_1. \end{cases}$$

易见 $d(s)$ 单调增,且 $x(s) \geqslant d(s)$, $s \leqslant t^*$.

若 (2) 成立,由 (1)

$$e^{\lambda t^*} x(t^*) = \int_{t_1}^{t^*} (e^{\lambda s} x(s))' ds = \int_{t_1}^{t^*} (e^{\lambda s} F(s, x_s)) ds$$

$$\leqslant \int_{t_1}^{t^*} e^{\lambda s} \int_{0}^{\infty} N_u(-x_s) d\_(u) ds$$

$$= \int_{t_1}^{t^*} e^{\lambda s} \int_{0}^{\infty} \max_{f \in [-u, 0]} (-x(s + f)) d\_(u) ds$$

$$\leqslant \int_{t_1}^{t^*} e^{\lambda s} \int_{0}^{\infty} \max_{f \in [-u, 0]} (-d(s + f)) d\_(u) ds$$

$$= \int_{t_1}^{t^*} \int_{0}^{\infty} e^{\lambda s} d(s - u) ds d\_(u)$$

$$= -\int_{0}^{\infty} \int_{t_1 - u}^{t^* - u} e^{\lambda(s + u)} d(s) ds d\_(u)$$

$$= -\int_0^{t^*-t_2}\int_{t_1-u}^{t^*-u} e^{\lambda(s+u)}\, d(s)\, ds\, d\_(u)$$

$$-\int_{t^*-t_2}^{+\infty}\int_{t_1-u}^{t^*-u} e^{\lambda(s+u)}\, d(s)\, ds\, d\_(u) = I_1 + I_2, \quad (15)$$

其中 $I_1 = -\int_0^{t^*-t_2}\int_{t_1-u}^{t^*-u} e^{\lambda(s+u)}\, d(s)\, ds\, d\_(u)$

$$= -\int_0^{t^*-t_2}\int_{t_2}^{t^*-u} e^{\lambda(s+u)}\, d(s)\, ds\, d\_(u)$$

$$-\int_0^{t^*-t_2}\int_{t_1-u}^{t_1} e^{\lambda(s+u)}\, d(s)\, ds\, d\_(u)$$

$$= I_{11} + I_{12}.$$

$$I_{11} = -X\int_0^{t^*-t_2}\int_{t_2}^{t^*-u} e^{\lambda u}\left(\frac{\_0}{\lambda}e^{\lambda s} - \left(\frac{\_0}{\lambda} - 1\right)e^{\lambda t^*}\right) ds$$

$$= -Xe^{\lambda t^*}\int_0^{t^*-t_2}\left(\frac{\_0}{\lambda^2}\left(1 - e^{-\lambda(t^*-t_2)}e^{\lambda u}\right) - \left(\frac{\_0}{\lambda} - 1\right)(t^* - t_2 - u)e^{\lambda u}\right) d\_(u)$$

$$= -Xe^{\lambda t^*}\int_0^{t^*-t_2}\left(\frac{\_0}{\lambda^2} - \left(\frac{\_0}{\lambda} - \frac{1}{\lambda}\right)e^{\lambda u} - \left(\frac{\_0}{\lambda} - 1\right)(t^* - t_2 - u)e^{\lambda u}\right) d\_(u)$$

$$I_{12} = X\int_0^{t^*-t_2}\int_{t_1-u}^{t_1} e^{\lambda(s+u)}\frac{\_0}{\lambda}\left(e^{\lambda(t_1-s)} - 1\right) ds\, d\_(u)$$

$$= e^{\lambda t^*}X\cdot\frac{\_0}{\lambda}e^{-\lambda(t^*-t_1)}\left(\left(u - \frac{1}{\lambda}\right)e^{\lambda u} + \frac{1}{\lambda}\right)$$

$$\leqslant e^{\lambda t^*}X\cdot\frac{\_0}{\lambda}e^{-\lambda(t^*-t_2)}\left(\left(u - \frac{1}{\lambda}\right)e^{\lambda u} + \frac{1}{\lambda}\right)$$

$$= e^{\lambda t^*}X\cdot\left(\frac{\_0}{\lambda} - 1\right)\left(ue^{\lambda u} - \frac{1}{\lambda}e^{\lambda u} + \frac{1}{\lambda}\right),$$

$$I_2 = X\int_{t^*-t_2}^{+\infty}\int_{t_1}^{t^*} e^{\lambda s}\min\left\{1, \frac{\_0}{\lambda}\left(e^{\lambda(t_1-s+u)} - 1\right)\right\} ds\, d\_(u)$$

$$\leqslant X\int_{t^*-t_2}^{+\infty}\left(\int_{t_1}^{t_2} e^{\lambda s}ds + \frac{\_0}{\lambda}\int_{t_2}^{t^*}\left(e^{\lambda(t_1+u)} - e^{\lambda s}\right)ds\right) d\_(u),$$

其中 $\int_{t_2}^{t^*}\left(e^{\lambda(t_1+u)} - e^{\lambda s}\right)ds$

$$= e^{\lambda u}\int_{t_2}^{t^*}\left(e^{\lambda s}(1 - e^{-\lambda u}) - (e^{\lambda s} - e^{\lambda t_1})\right)ds$$

$$\leqslant e^{\lambda u}\int_{t_2}^{t^*}\left(e^{\lambda s}(1 - e^{-\lambda u}) - e^{-\lambda t^*}\cdot e^{\lambda s}(e^{\lambda s} - e^{\lambda t_1})\right)ds$$

$$= e^{\lambda u}\left\{\frac{1 - e^{-\lambda u}}{\lambda}\left(e^{\lambda t^*} - e^{\lambda t_2}\right)\right.$$

$$\left. - e^{-\lambda t^*}\left(\frac{1}{2\lambda}\left(e^{2\lambda t^*} - e^{2\lambda t_2}\right) - \frac{e^{\lambda t_1}}{\lambda}\left(e^{\lambda t^*} - e^{\lambda t_2}\right)\right)\right\}$$

$$= e^{\lambda t^*}\left\{\frac{e^{\lambda u} - 1}{\lambda}\left(1 - e^{-\lambda(t^*-t_2)}\right) - \frac{1}{2\lambda}\left(1 - e^{-\lambda(t^*-t_2)}\right)^2\right\}$$

$$- \frac{e^{\lambda u}}{\lambda}\left(e^{\lambda t_2} - e^{\lambda t}\right)\left(1 - e^{-\lambda(t^*-t_2)}\right),$$

从而

$$I_2 \leqslant X\int_{t^*-t_2}^{+\infty}\frac{1}{\lambda}\left(e^{\lambda t_2} - e^{\lambda t_1}\right)\left(1 - e^{\lambda u}\right) d\_(u) + e^{\lambda t^*}\cdot X\int_{t^*-t_2}^{+\infty}\left(\frac{e^{\lambda u} - 1}{\lambda} - \frac{e^{\lambda u}}{2\_0}\right) d\_(u) \leqslant e^{\lambda t^*}X\int_{t^*-t_2}^{+\infty}\left(\frac{e^{\lambda u} - 1}{\lambda} - \frac{e^{\lambda}}{2\_0}\right) d\_(u),$$

由(15)、(6)及对 I_{11}、I_{13}、I_2 的估计,

$$e^{\lambda t^*}x(t^*) \leqslant e^{\lambda t^*}X\int_0^{+\infty}\left(\frac{e^{\lambda u} - 1}{\lambda} - \frac{e^{\lambda}}{2\_0}\right) d\_(u)$$

$$- e^{\lambda t^*}X\int_0^{t^*-t_2}\left(\frac{e^{\lambda u}}{\lambda} - \frac{1}{2\_0}e^{\lambda u}\right.$$

$$\left. + \frac{e^{\lambda u}}{\lambda}\left(\frac{\_0}{\lambda} - 1\right)\ln\left(1 - \frac{\lambda}{\_0}\right)\right) d\_(u)$$

$$= Xe^{\lambda t^*}\left(\frac{1 - \_0}{\lambda} - \frac{1}{2\_0} - \frac{\_0}{\lambda^2}f\left(\frac{\lambda}{\_0}\right)\right.$$

$$\left.\int_0^{-\frac{1}{\lambda}\ln\left(1 - \frac{\lambda}{\_0}\right)} e^{\lambda u} d\_(u)\right) \leqslant Xe^{\lambda t^*}.$$

这与 $x(t^*) > X$ 矛盾.

若(3)成立,则易证 $e^{\lambda t^*}x(t^*) \leqslant I_{12} + I_2 = I_1 + I_2 + (-I_{11})$.

由前面的证明,

$$e^{\lambda t^*}x(t^*) \leqslant Xe^{\lambda t^*}\left(\frac{1 - \_0}{\lambda} - \frac{1}{2\_0} - \_2\right) + (-I_{11}). \quad (16)$$

下面估计 $(-I_{11})$,

$$-I_{11} = Xe^{\lambda t^*}\int_0^{t^*-t_2}\left(\frac{\_0}{\lambda_2} - \left(\frac{\_0}{\lambda} - 1\right)e^{\lambda u}\left(\frac{1}{\lambda} + t^* - t_2 - u\right)\right) d\_(u)$$

$$\leqslant Xe^{\lambda t^*}\int_0^{t^*-t_2}\left(\frac{\_0}{\lambda_2} - \left(\frac{\_0}{\lambda} - 1\right)\left(\frac{1}{\lambda} + t^* - t_2\right)\right) d\_(u)$$

$$= Xe^{\lambda t^*}\frac{\_0}{\lambda_2}\left(\frac{\lambda}{\_0} + \left(1 - \frac{\lambda}{\_0}\right)\ln\left(1 - \frac{\lambda}{\_0}\right)\right)\int_0^{t^*-t_2} d\_(u)$$

$$= Xe^{\lambda t^*}\_3.$$

于是由(16)、(8)得

$$e^{\lambda t^*}x(t^*) \leqslant Xe^{\lambda t^*}\left(\frac{1 - \_0}{\lambda} - \frac{1}{2\_0} - \_2 + \_3\right) \leqslant Xe^{\lambda t^*}.$$

这与 $x(t^*) > X$ 矛盾.证毕.

利用引理 1,可以证明在条件(2)、(6)或者(3)、(8)下,(1)的零解一致稳定.

定理 1与定理 2(i)的证明:

证明 记 $h = \frac{1}{\lambda}\ln\left(1 + \frac{\lambda}{\_0}\right)$, $H_0 = \exp\left(\_0 h + \int_0^{\infty} u\, d\_(u)\right)$,取 W使得 $WH_0 \leqslant H$, $\|x_{t_0}\| \leqslant W$.由(1)、(2)或(3),

$$|x(t)| \leqslant e^{-\lambda(t-t_0)}|x(t_0)| + \int_{t_0}^{t} e^{-\lambda(t-s)}\|x_s\|\, ds\cdot\_0$$

$$\leqslant |x(t_0)| + \int_{t_0}^{t}\|x_s\|\, ds\cdot\_0, \quad t \geqslant t_0.$$

从而

$$\| x_t \| \leqslant \| x_{t_0} \| + \int_{t_0}^{t} \| x_s \| \, ds \cdot \_{}_0, t \geqslant t_0.$$

利用 Grownwall 不等式

$$\| x_t \| \leqslant \| x_{t_0} \| \exp(\_(t - t_0)), t \geqslant t_0,$$

从而当 $t \in [t_0, t_0 + h]$ 时,

$$\| x_t \| \leqslant \| x_{t_0} \| \exp(\_{}_0 h) \leqslant W \exp(\_{}_0 h), \quad (17)$$

下证当 $t \geqslant t_0 + h$ 时,

$$D^+ \| x_t \| \leqslant \int_{t-(t_0+h)}^{+\infty} d\_(u) \cdot \| x_t \| . \quad (18)$$

实际上,若 $|x(t)| < \| x_t \|$,则 $D^+ \| x_t \| = 0$. 若 $|x(t)| = \| x_t \|$,则 $D^+ \| x_t \| = D^+ |x(t)| \geqslant 0$, 不妨设 $x(t) > 0$,若 $x'(t) = 0$,则 (22) 成立. 下设 $x'(t) > 0$. 由引理 1, $x(s) > 0, s \in [t_0 + h, h]$. 于是

$$(e^{\lambda t} x(t))' = e^{\lambda t} F(t, x_t)$$

$$\leqslant e^{\lambda t} \int_0^{\infty} \max\left\{0, \max_{f \in [-u, 0]} (-x(t + f))\right\} du$$

$$= e^{\lambda t} \left(\int_0^{t-(t_0+h)} + \int_{t-(t_0+h)}^{+\infty} \right) \max\left\{0, \max_{f \in [-u, 0]} (-x(t + f))\right\} d\_(u)$$

$$\leqslant e^{\lambda t} \int_{t-(t_0+h)}^{+\infty} d\_(u) \cdot \| x_t \| ,$$

从而

$$x'(t) \leqslant \int_{t-(t_0+h)}^{+\infty} d\_(u) \cdot \| x_t \| - \lambda x(t) \leqslant \int_{t-(t_0+h)}^{+\infty} d\_(u) \cdot \| x_t \| ,$$

故 (21) 总成立,由此得到

$$\| x_t \| \leqslant \| x_{t_0+h} \| \exp\left(\int_{t_0+h}^{t} \int_{s-(t_0+h)}^{+\infty} d\_(u) ds \right) \leqslant \| x_{t_0+h} \| \exp\left(\int_0^{+\infty} \int_{t_0+h}^{t_0+h+u} ds\, d\_(u) \right) \leqslant \| x_{t_0} \| \exp\left(\_{}_0 + h + \int_0^{\infty} u\, d\_(u) \right) \leqslant W H, t \geqslant t_0 + h. \quad (19)$$

由 (17)、(19) 易知 (1) 的零解一致稳定. 证毕.

为了得到 (1) 的零解的一致渐过稳定性,还需利用下面两个引理.

引理 2 设 (2)、(6) 成立或者 (3)、(8) 成立. 又设 $W_0 > 0, X > 0$,满足 $W_0 H_0 \leqslant H, X_0 < W_0 H_0$. 则对此 W_0, X_0,存在 $T > 0$,使得对 (1) 任一满足 $\| x_{t_0} \| \leqslant W_0$ 的解 $x(t)$,当 $t \geqslant t_0$, $|x(t + T)| \geqslant X_0$ 蕴含着存在 $f \in [t, t + T)$ 使得 $x(f) = 0$,而 $|x(s)| > 0, s \in (f, t + T]$.

引理 2 的证明类似于文献 [5] 引理 2.2 的证明,此处从略.

引理 3 设 (2)、(7) 成立或者 (3)、(9) 成立. 又设 $W_0 > 0, X_0 > 0$,满足 $W_0 H_0 \leqslant H, X_0 < W_0 H_0$. 则对此 W_0, X_0,存在 $T_0 > 0, U_0 > 0$ 使得对任意 $T \in (0, T_0]$ 及任意 $U \geqslant U_0$,由

$$\| x_{t_0} \| \leqslant W_0, t_1 \geqslant t_0, t \geqslant t_1 + 2U, |x(t)| \geqslant X_0,$$

$$|x(s)| \leqslant (1 + T) |x(t)|, s \in [t_1, t],$$

可推出 $|x(s)| > 0, s \in [t_1 + 2U, t]$.

证明 不妨设 $x(t) > 0$. 若上述论断不成立,则存在序列 $\{T_n\}$, $\{U_n\}$, $T_n \to 0$, $U_n \to +\infty$,及序列 $\{t_n\}$, $\{s_n\}$, $\{S_n^1\}$, $\{T_n\}$,解序列 $\{x^n\}$ 使得 $\| x_{t_n}^n \| \leqslant W_0, s_n \geqslant t_n, T_n > 0, S_n^1 \geqslant s_n + 2U_n, x^n(S_n^1) = 0, x^n(s) = 0, s \in (S_n^1, S_n^1 + T_n], |x^n(s)| \leqslant (1 + T_n) x^n(s_n + 2U_n + T_n), s \in [s_n, S_n^1 + T_n]$,而 $x^n(S_n^1 + T_n) \geqslant X_0$. 由引理 2, $T_n \leqslant T$. 下记 $S_n^3 = S_n^1 + T_n$, $X = x^n(S_n^3)$. 由一致稳定性的证明,当 $t \geqslant t_n$ 时, $|x^n(t)| \leqslant W_0 H_0$,由 (1)、(2) 或 (3),当 $t \in [s_n + U_n, S_n^3]$,

$$\frac{d}{dt}(e^{\lambda t} x^n(t)) = e^{\lambda t} F(t, x_t^n)$$

$$\leqslant e^{\lambda t} \int_0^{\infty} \max\left\{0, \max_{s \in [-u, 0]} (-x(t + s))\right\} d\_(u)$$

$$= e^{\lambda t} \left(\int_0^{U_n} + \int_{U_n}^{+\infty} \right) \max\{0, \max_{s \in [-u, 0]} (-x(t + s))\} d\_(u)$$

$$\leqslant e^{\lambda t} \left[(1 + T_n) X \int_0^{U_n} d\_(u) + W_0 H_0 \int_{U_n}^{\infty} d\_(u) \right]$$

$$\leqslant v_n \_{}_0 e^{\lambda t} X,$$

其中

$$v_n = \frac{1 + T_n}{\_{}_0} \left[\int_0^{U_n} d\_(u) + \frac{W_0 H_0}{X_0} \int_{U_n}^{\infty} d\_(u) \right],$$

显然 $v_n > 1$ 且 $v_n \to 1$,易见,当 $t \leqslant S_n^3$ 时,

$$x^n(t) \geqslant X \left[\frac{v_n \_{}_0}{\lambda} - \left(\frac{v_n \_{}_0}{\lambda} - 1 \right) e^{\lambda (S_n^3 - t)} \right], \quad (20)$$

若 $\lambda > \_{}_0$,由于 $v_n \to 1$,故对充分大的 n 有 $\frac{v_n \_{}_0}{\lambda} < 1$, 从而,对此 n,

$$x^n(S_n^1) \geqslant X \left[\frac{v_n \_{}_0}{\lambda} + \left(1 - \frac{v_n \_{}_0}{\lambda} \right) e^{\lambda (S_n^3 - t)} \right] > X \frac{v_n \_{}_0}{\lambda},$$ 与 $x^n(S_n^1) = 0$ 矛盾,故以下设 $\lambda \leqslant \_{}_0$,取 S_n^2 使得

$$e^{-\lambda (S_n^3 - S_n^2)} = 1 - \frac{\lambda}{v_n \_{}_0},$$ 由 $x^n(S_n^1) = 0$ 及 (20) 可知 $S_n^2 \geqslant S_n^1$,定义

$$d^n(s) = \begin{cases} X \left[\frac{v_n \_{}_0}{\lambda} - \left(\frac{v_n \_{}_0}{\lambda} - 1 \right) e^{\lambda (S_n^3 - s)} \right], & s \in [S_n^2, S_n^3] \\ 0, & s \in [S_n^1, S_n^2] \\ -v^n X \min\{1, \frac{\_{}_0}{\lambda} (e^{\lambda (S_n^3 - s)} - 1)\}, & s \leqslant S_n^1. \end{cases}$$

易证 $d^n(s)$ 单调增,且 $x^n(s) \geqslant d^n(s), s \leqslant S_n^3$.

若 (2) 成立,由 (1)、(2),

$$e^{\lambda S_n^3} X \leqslant -\int_0^{\infty} \int_{S_n^1 - u}^{S_n^3 - u} e^{\lambda (s + u)} d^n(s) ds\, d\_(u) = -\left(\int_0^{2U_n} \right.$$

$$+\int_{2U_n}^{\infty}\Big(\int_{S_n^1-u}^{S_n^3-u} e^{\lambda(s+u)} d^t(s)\,ds\,d\_(u)$$

$$\leqslant -\int_0^{2U_n}\int_{S_n^1-u}^{S_n^3-u} e^{\lambda(s+u)} d^t(s)\,ds\,d\_(u)+\frac{H_0W_0}{\lambda}e^{\lambda S_n^3}(1-$$

$$e^{-\lambda(S_n^3-S_n^1)}\Big)\int_{2U_n}^{+\infty} d\_(u).$$

由于 $U_n\to\infty$,故对充分大的 n,有 $2U_n\geqslant T\geqslant S_n^3-S_n^1$,于是,对此 n,

$$e^{\lambda S_n^3}X\leqslant -\Big(\int_0^{S_n^3-S_n^2}+$$

$$\int_{S_n^3-S_n^2}^{2U_n}\Big)\int_{S_n^1-u}^{S_n^3-u} e^{\lambda(s+u)} d(s)\,ds\,d\_(u)+\frac{H_0W_0}{\lambda}\int_{2U_n}^{+\infty} d\_(u)$$

$$\leqslant -\Big(\int_0^{S_n^3-S_n^2}+\int_{S_n^3-S_n^2}^{+\infty}\Big)\int_{S_n^1-u}^{S_n^3-u} e^{\lambda(s+u)} d(s)\,ds\,d\_(u)+$$

$$W_n e^{\lambda S_n^3},$$

其中 $W_n=\frac{H_0W_0}{\lambda}\int_{2U_n}^{+\infty} d\_(u)$,易见当 $n\to\infty$ 时,$W_n\to 0$.

类似于引理 1的证明,

$$e^{\lambda S_n^3}X\leqslant e^{\lambda S_n^3}X\int_0^{+\infty}\Big(\frac{e^{\lambda u}-1}{\lambda}-\frac{e^{\lambda u}}{2v_{n}\_0}\Big)d\_(u)-$$

$$e^{\lambda S_n^3}X\cdot\frac{\_0}{\lambda^2}\Big\{\frac{\lambda}{\_0}-\frac{\lambda^2}{2v_n\_0^2}+(v_n-\frac{\lambda}{\_0}\ln(1-\frac{\lambda}{v_n\_0}))\Big\}$$

$$\int_0^{-\frac{1}{\lambda}\ln(1-\frac{\lambda}{v_n\_0})} e^{\lambda u} d\_(u)+W_n e^{\lambda S_n^3},$$

从而

$$1\leqslant\int_0^{+\infty}\Big(\frac{e^{\lambda u}-1}{\lambda}-\frac{e^{\lambda u}}{2v_n\_0}\Big)d\_(u)-\frac{\_0}{\lambda^2}\Big\{\frac{\lambda}{\_0}-\frac{\lambda^2}{2v_n\_0^2}+(v_n-\frac{\lambda}{\_0}\ln(1-\frac{\lambda}{v_n\_0}))\Big\}\cdot$$

$$\int_0^{-\frac{1}{\lambda}\ln(1-\frac{\lambda}{v_n\_0})} e^{\lambda u} d\_(u)+\frac{W_n}{X}. \qquad (21)$$

若 $\lambda<\_0$,令 $n\to\infty$,由 $v_n\to\dagger$,$W_n\to 0$及 (7)得

$$1\leqslant\int_0^{+\infty}\Big(\frac{e^{\lambda u}-1}{\lambda}-\frac{e^{\lambda u}}{2\_0}\Big)d\_(u)-\frac{\_0}{\lambda^2}f\Big(\frac{\lambda}{\_0}\Big)$$

$$\int_0^{-\frac{1}{\lambda}\ln(1-\frac{\lambda}{\_0})} e^{\lambda u}d\_(u)=\frac{1-\_0}{\lambda}-\frac{1}{2\_0}-\_^2<1.$$

矛盾.

若 $\lambda=\_0$, (21) 变为

$$1\leqslant\int_0^{+\infty}\Big(\frac{e^{\lambda u}-1}{\lambda}-\frac{e^{\lambda u}}{2v_n\lambda}\Big)d\_(u)-\frac{1}{\lambda}\Big\{1-\frac{1}{2v_n}+$$

$$(v_n-1)\ln(1-\frac{1}{v_n})\Big\}\int_0^{-\frac{1}{\lambda}\ln(1-\frac{1}{v_n})} e^{\lambda u} d\_(u)+\frac{W_n}{X},$$

令 $n\to\infty$,由 $v_n\to\dagger$,$W_n\to 0$得

$$1\leqslant\int_0^{\infty}\Big(\frac{e^{\lambda u}-1}{\lambda}-\frac{e^{\lambda u}}{2\lambda}\Big)d\_(u)-\frac{1}{2\lambda}\int_0^{\infty}e^{\lambda u}d\_(u)$$

$$=-\frac{1}{\lambda}\int_0^{\infty}d\_(u).$$

矛盾.

类似地可以证明当 (3)、(9) 成立时,也可导出矛盾.证毕.

定理 1与定理 2(ii) 的证明:

证明　由一致稳定性的证明,若 W_0满足 $W_0H_0\leqslant H$,则当 $\|x_{t_0}\|\leqslant W_0$时,

$$\|x_t\|\leqslant\|x_{t_0}\|\cdot H_0, \qquad (22)$$

下证在此 W_0下, (1) 的零解一致吸引.若不然,存在 $X_0\in(0,W_0H_0)$, $\{t_n\}$、$\{T_n\}$及 (1) 的解序列 $\{x^n\}$,使得 $t_n\geqslant 0$, $T_n\to\infty$, $\|x_{t_n}\|<W_0$,而 $|x^n(t_n+T_n)|\geqslant X_0$.对此 W_0, X_0,可得到保证引理 2成立的常数 $T>0$及保证引理 3成立的常数 $T_0>0$, $U_0>0$.现取正整数 k, n使得 $(1+T_0)^kX_0>W_0H_0$, $T_n>k(T+2U_0)$.记 $f_0=t_n+T_n$.则 $|x^n(f_0)|\geqslant X_0$,由引理 2,存在 $t\in[f_0-T,f_0]$,使得 $x^n(t)=0$.又由引理 3,存在 $f_1\in[f_0-(T+2U_0),f_0)$使得 $|x^n(f_1)|>(1+T_0)|x^n(f_0)|$.类似地,存在 $f_2\in[f_1-(T+2U_0),f_1)$满足 $|x^n(f_2)|>(1+T_0)|x^n(f_1)|$.重复上述步骤,存在 $f_k\in[f_{k-1}-(T+2U_0),f_{k-1})\subset[t_n,t_n+T_n]$使得 $|x^n(f_k)|>(1+T_0)|x^n(f_{k-1})|>(1+T_0)^k|x^n(f_0)|\geqslant(1+T_0)^kX_0>W_0H_0$,这与 (22) 矛盾.证毕.

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