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Another Tree in Symbolic Dynamics Being Isomorphic to Almeida-Ramos′ Tree*
符号动力系统中同构于 Almeida-Ramos树的另一个树

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Abstract　On the basis of the pure combinatorial procedure a tree of 0-1 sequences is constructed, which is shown to be isomorphic to the tree constructed by Almeida-Ramos. Also self-similiarity and eigenvalues of elememts in such a tree are studied.

Key words　symbolic dynamical systems, isomorphism of trees, self-similarity, eigenvalue

摘要　运用纯组合学方法构造了 0-1序列的一个树 T,证明了 T同构于 Almeida-Ramos树.此外,还研究了 T的自相似性及 T中元素的特征值.

关键词　符号动力系统　树的同构　自相似性　特征值
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1 Instruction

Almeida-Romos in reference [1] constructed a tree of 0-1 sequences, written T_{AR}, by using a pure combinatorial procedure and studied the norms of matrices induced by elements of T_{AR}. In this paper we construct, in a similar way, a tree, written T, of 0-1 sequences which is different from T_{AR} but shown to be isomorphic it. Moreover, we study the self-similarity in T and the distribution of eigenvalues of elements of T. According to reference [2], the tree T is closely related to the dynamics of Lorenz maps.

2 Another Tree in Symbolic Dynamics

Let $\sum = \prod_1^\infty \{0,1\}$ and $\sigma: \sum \to \sum$ be defined by $\sigma(x_1,x_2,\cdots) = (x_2,x_3,\cdots)$ for any $(x_1,x_2,\cdots)\in\sum$. The $(\sum,\sigma)$ (with the usual product topology induced by the discrete topology of {0, 1}) is called to be the (one-sided) symbolic dynamical system (see reference [3] for more details).

Let $x,y\in\sum$ be two periodic sequences of periods n and m, respectively. Put $x = (x_1,x_2,\cdots,x_n, x_1,x_2,\cdots,x_n,\cdots)$ and $y = (y_1,y_2,\cdots,y_m,y_1,y_2,\cdots,y_m,\cdots)$. We identify x with y if $n = m$ and $x_1x_2\cdots x_n = y_1y_2\cdots y_m$. with this identification, denote by $\overline{\sum}$ the related identification space, and by $x_1x_2,\cdots,x_{n-1}\triangle$ the identification class of the periodic sequence x of period n in $\overline{\sum}$, where "$\triangle$" is a symbol different from either 0 or 1.

Let $1 < \triangle < 0$, define an order $<$ in $\overline{\sum}$ as follows for $x = x_1x_2,\cdots, y = y_1y_2,\cdots\in\overline{\sum}$ with $x \neq y$, we say that $x < y$ if there exists an index i such that

$$x_1x_2,\cdots,x_{i-1} = y_1y_2,\cdots,y_{i-1} \text{ and } x_i < y_i.$$

A sequence $x\in\overline{\sum}$ is said to be shift-maximal if, for any $1\leqslant k\leqslant |x| - 1$, $\max\{\sigma^k x, \sigma^k x'\}\leqslant x$, where $|x|$ denote the length of x, $x' = x'_1x'_2,\cdots$, $0' = 1$, $1' = 0$ and $\triangle' = \triangle$. Denote by $\sum^*$ the set of shift-maximal sequences in $\overline{\sum}$ and by $\sum_k$ the set of shift-maximal sequences with length k for $k\geqslant 1$. Then we have

$$\sum{}^* = \overline{\bigcup_{k=1}^\infty \sum{}_k}$$

for the topology of total order in $\overline{\sum}$ induced by the total order $<$.

We figure the element of $\sum^*$ as a tree, written T in what follows, placing at the k-level the

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elements of $\sum_k$ in increasing order from left to right, The Fig. 1 below illustrates what we mean.

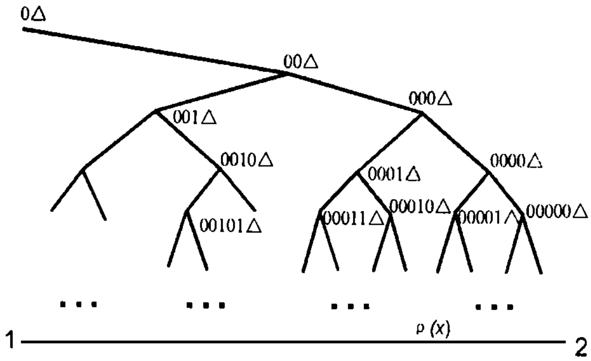

Fig. 1 A tree T satisfying $\overline{\delta(T)} \equiv \overline{\{\delta(x): x\in T\}} = \overline{\{\delta(x): x\in T\cap P\}} = [1,2]$.

Denote by $\sum_{\rm fin}$ the set of finite 0 – 1 sequences. Define two maps φ and ψ as follows:

$$\varphi: \sum_{\rm fin}\cup \overline{\sum} \to \sum_{\rm fin}\cup\overline{\sum},$$

$$x = x_1x_2\cdots \mapsto a = a_1a_2\cdots,$$

where $a_1 = x_1$ and, for $2\leqslant i\leqslant |x|$,

$$a_i = \begin{cases} x'_i, & \text{if } x_{i-1} = 0,\\ x_i, & \text{if } x_{i-1} = 1,\end{cases}$$

and

$$\psi: \sum_{\rm fin}\cup \overline{\sum} \to \sum_{\rm fin}\cup\overline{\sum},$$

$$y = y_1y_2\cdots \mapsto b = b_1b_2\cdots,$$

where $b_1 = y_1$ and, for $2\leqslant j\leqslant |y|$,

$$b_j = \begin{cases} y'_j, & \text{if the parity of } 0'\text{s in } y_1y_2\cdots y_{j-1} \text{ is odd},\\ y_j, & \text{if the parity of } 0'\text{s in } y_1y_2\cdots y_{j-1} \text{ is even}.\end{cases}$$

Clearly, both φ and ψ are injective.

Lemma 2.1 (i) Let $x = x_1x_2,\cdots\in\sum_{\rm fin}\cup\overline{\sum}$ and $a = \varphi(x) = a_1a_2\cdots$. Then, for any $1\leqslant j\leqslant |a|$, the parity of $0'$s in $a_1a_2\cdots a_j$ is odd (even, resp.) if and only if $x_j = 0$ ($x_j = 1$, resp.).

(ii) $\psi\circ\varphi = \varphi\circ\psi = id_{\Sigma_{\rm fin}\cup\overline{\Sigma}}$.

Proof (i) By induction on j. Clearly it is true for $j = 1$. Now suppose $j\geqslant 2$ and the conclusion is true for $j-1$. Without loss of generality, assume the parity of $0'$s in $a_1a_2\cdots a_{j-1}$ is odd and $x_{j-1} = 0$. If the parity $0'$s in $a_1a_2\cdots a_j$ is odd, then $a_j = 1$ and thus $x_j = a'_j = 0$, conversely, if $x_j = 0$, then $a_j = x'_j = 1$ and thus the parity of $0'$s in $a_1a_2\cdots a_{j-1}$ is odd; if the parity of $0'$s in $a_1a_2\cdots a_j$ is even, then $a_j = 0$ and thus $x_j = a'_j = 1$, conversely, if $x_j = 1$, then $a_j = x'_j = 0$ and thus the parity of $0'$s in $a_1a_2\cdots a_j$ is even.

(ii) We prove only $\psi\circ\varphi = id_{\Sigma_{\rm fin}\cup\overline{\Sigma}}$, the a_j proof of that $\psi\circ\varphi = id_{\Sigma_{\rm fin}\cup\overline{\Sigma}}$ is similar and omitted.

Let $x = x_1x_2\cdots\in\sum_{\rm fin}\cup\overline{\sum}$, $a = a_1a_2\cdots = \varphi(x)$ and $y = y_1y_2\cdots = \psi(a)$. By the definitions of φ and ψ, we have $y_1 = a_1 = x_1$, and, for any $2\leqslant i\leqslant |x|$,

$$y_i = \begin{cases} a'_i, & \text{if the parity of } 0'\text{s in } a_1a_2\cdots a_{i-1} \text{ is odd},\\ a_i, & \text{if the parity of } 0'\text{s in } a_1a_2\cdots a_{i-1} \text{ is even},\end{cases}$$

where

$$a_i = \begin{cases} x'_i, & \text{if } x_{i-1} = 0,\\ x_i, & \text{if } x_{i-1} = 1.\end{cases}$$

If $x_i = \triangle$ then $y_i = a_i = \triangle = x_i$. Now suppose $x_i\neq\triangle$. Then $a_i\neq\triangle$ and $y_i\neq\triangle$. By (i), thus, one gets $y_i = a_i = x_i$. This completes the proof.

Let T_{AR} be the tree of 0–1 sequences constructed by Almeida–Ramos in reference [1]. In what follows we identify $c_1c_2\cdots c_i\triangle$ with $c_1c_2\cdots c_i^*$ for any $c_1c_2\cdots c_i^*\in T_{AR}$.

Theorem 2.2 T is isomorphic to T_{AR}.

Proof For convenience, denote by φ and ψ the restrictions of φ and ψ to T and T_{AR}, respectively. We will prove that $\varphi: T\to T_{AR}$ is an isomorphism and complete the proof in following two steps.

Step 1 To prove that both $\varphi: T\to T_{AR}$ and $\psi: T_{AR}\to T$ are well defined.

We prove only that $\varphi: T\to T_{AR}$ is well defined, the proof of that $\psi: T_{AR}\to T$ is well defined is similar and omitted. Clearly it suffices to prove that $\varphi(x)\in T_{AR}$ for any $x\in T$, that is, to prove that $\sigma^k(\varphi(x))\leqslant\varphi(x)$ for any $1\leqslant k\leqslant |x| - 1$.

Let $x = x_1x_2\cdots x_{n-1}\triangle$ and $\varphi(x) = a = a_1a_2\cdots a_{n-1}\triangle$. Set $\max\{\sigma^k(x), \sigma^k(x')\} = y_{k+1}y_{k+2}\cdots y_{n-1}y_n$, where $y_n = \triangle$. Then we have

$$y_{k+1}y_{k+2}\cdots y_{n-1}y_n\leqslant x.$$

Assume

$$y_{k+1}y_{k+2}\cdots y_{k+r-1} = x_1x_2\cdots x_{r-1} \text{ and } y_{k+r} < x_r$$

for some $1\leqslant r\leqslant n-k$. Then we do argument in following four cases

Case 1 $y_{k+1}y_{k+2}\cdots y_{k+r-1} = x_{k+1}x_{k+2}\cdots x_{k+r-1}$ and $x_k = 0$. Then $a_{k+1} = x'_{k+1} = y'_{k+1} = x'_1 = 1 < 0 = a_1$. This implies $\sigma^k a < a$.

Case 2 $y_{k+1}y_{k+2}\cdots y_{k+r-1} = x'_{k+1}x'_{k+2}\cdots x'_{k+r-1}$ and $x_k = 1$. Then $a_{k+1} = x_{k+1} = y'_{k+1} = x'_1 = 1 < 0 = a_1$. This implies $\sigma^k a < a$.

Case 3 $y_{k+1}y_{k+2}\cdots y_{k+r-1} = x'_{k+1}x'_{k+2}\cdots x'_{k+r-1}$ and $x_k = 0$. Then $a_{k+1} = x'_{k+1} = y_{k+1} = x_1 = 0$. For any $2\leqslant i\leqslant r-1$, if $x_{k+i-1} = 0$, then $x_{i-1} = y_{k+i-1} = x'_{k+i-1} = 1$ and thus we have $a_{k+i} = x'_{k+i} = y_{k+i} = x_i = a_i$; if $x_{k+i-1} = 1$, then $x_{i-1} = x'_{k+i-1} = 0$ and thus we have $a_{k+i} = x_{k+i} = y_{k+i} = x'_i = a_i$. That is, in this case, we have

$$a_1a_2\cdots a_{r-1} = a_{k+1}a_{k+2}\cdots a_{k+r-1}.$$

If $x_{k+r-1} = 0$, then $x_{r-1} = y_{k+r-1} = x'_{k+r-1} = 1$ and thus we have $a_{k+r} = x'_{k+r} = y_{k+r} < x_r = a_r$. Note that in this case the parity of $0's$ in $a_1a_2\cdots a_{i-1}$ is even. Thus one gets $\sigma^k a < a$; if $x_{k+r-1} = 1$, then a similar argument implies $\sigma^k a < a$.

Case 4 $y_{k+1}y_{k+2}\cdots y_{k+r-1} = x_{k+1}x_{k+2}\cdots x_{k+r-1}$ and $x_k = 1$. Obviously, we have, in this case, $a_{k+1}a_{k+2}\cdots a_{k+r-1} = a_1a_2\cdots a_{r-1}$ and $a_{k+r}\neq a_r$. If $x_{k+r-1} = 0$, then $x_{r-1} = y_{k+r-1} = x_{k+r-1} = 0$ and thus we have $a_{k+r} = x'_{k+r} = y'_{k+r} > x'_r = a_r$. Note that in this case the parity of $0's$ in $a_1a_2\cdots a_{i-1}$ is odd. Thus one gets $\tilde{a} < a$; if $x_{k+r-1} = 1$, then a similar argument implies $\tilde{a} < a$.

Step 2 To prove h is an order-preserving map.

Let $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_m$ be in T with $x < y$, and let $h(x) = a_1a_2\cdots a_n$ and $h(y) = b_1b_2\cdots b_m$, where $x_n = y_m = a_n = b_m = \triangle$. Assume

$$x_1x_2\cdots x_{i-1} = y_1y_2\cdots y_{i-1} \text{ and } x_i < y_i$$

for some $0 \leqslant i \leqslant \min\{n-1, m-1\}$. Then we have

$$a_1a_2\cdots a_{i-1} = b_1b_2\cdots b_{i-1} \text{ and } a_i \neq b_i.$$

If $x_{i-1} = 0$, then $y_{i-1} = 0$ and the parity of $0'$s in $a_1a_2\cdots a_{i-1}$ is odd. Thus we have $a_i = x'_i > y'_i = b_i$ and then $h(x) < h(y)$; if $x_{i-1} = 1$, then a similar argument implies $h(x) < h(y)$. Hence the map $h: T \to T_{AR}$ is order-preserving. For any $x, y \in T$, put $a = h(x)$ and $b = j(y)$, then there exists an edge between x and y in T if and only if there exists an edge between a and b in T_{AR}. Therefore, from Lemma 2.1 it follows that T is isomorphic to T_{AR} with respect to h. This completes the proof.

Remark 2.3 Denote by $\sum_{AR}$ the set $\sum$ in reference [1]. Clearly from the proof of Theorem 2.2 it follows that $(j|\sum_{AR}) \circ (h|\sum) = id_{\Sigma}$ and $(h|\sum) \circ (j|\sum_{AR}) = id_{\Sigma_{AR}}$.

3 Self-Similarity

In this section we study the self-similarity of T. The ideal of using self-similarity is motivated by reference [4].

Definition 3.1 A map $S: T \to T$ is said to be a self-similarity map if

(i) S is monotone increasing;

(ii) S has the following intermediate value property: Let T and U be in T with $T < U$. Then for any $\Gamma \in T$ satisfying $S(T) < \Gamma < S(U)$, there exists a $V \in T$ with $T < V < U$ such that $S(V) = \Gamma$.

Definition 3.2 Let $x = x_1x_2\cdots x_{n-1}\triangle \in T$. Define a map $S_x: \sum \to \sum$ by, for any $y = y_1y_2\cdots \in \sum$,

$$S_x(y) = x_1x_2\cdots x_{n-1}y_1\overline{y_1}y_2\overline{y_2}\cdots,$$

where, for any $1 \leqslant i \leqslant |y| - 1$,

$$\overline{y_i} = \begin{cases} x'_1x'_2\cdots x'_{n-1}, & \text{if } y_i = 0, \\ x_1x_2\cdots x_{n-1}, & \text{if } y_i = 1. \end{cases}$$

The map $S_x: \sum \to \sum$

is well defined by Lemma 3.1 below. Denote by M the set of shift-maximal sequences in reference [5]. Define two maps ξ and η as follows

$$a: \sum_{AR} \to M$$
$$a_1a_2\cdots \mapsto \check{a}_1\check{a}_2\cdots,$$

and

$$Z: M \to \sum_{AR}$$
$$A_1A_2\cdots \mapsto \check{A}_1\check{A}_2\cdots,$$

where $\check{0} = R$, $\check{R} = 0$, $\check{1} = L$, $\check{L} = 1$, $\check{\triangle} = C$ and $\check{C} = \triangle$. Clearly, $a \circ Z = id_M$, $Z \circ a = id_{\Sigma_{AR}}$ and both a and Z are order-preserving.

Lemma 3.1 Let $x \in T$. Then, for any $y \in \sum$,

$$S_x(y) = j(Z(a(h(x)) * a(h(y)))),$$

where $*$ -product is defined on page 72 in reference [3].

Proof Set

$$h(x) \otimes h(y) = Z(a(h(x)) * a(h(y))).$$

Then it suffices to prove

$$S_x(y) = j(h(x) \otimes h(y)).$$

Let $h(x) = a_1a_2\cdots a_{n-1}\triangle$ and $h(y) = b_1b_2\cdots$, put $a = a_1a_2\cdots a_{n-1}$. It is not difficult to obtain that

$$h(y) \otimes h(y) = a\bar{b}_1 a\bar{b}_2\cdots,$$

where

$$\bar{b}_i = \begin{cases} b'_i, & \text{if the parity of } 0's \text{ in } a \text{ is odd}, \\ b_i, & \text{if the parity of } 0's \text{ in } a \text{ is even}. \end{cases}$$

Note that the parity of $0'$s in a is odd (even, resp.) if and only if $x_{n-1} = 0$ ($x_{n-1} = 1$, resp.),

$$h(0x'_1x'_2\cdots x'_{n-1}) = 0h(x_1x_2\cdots x_{n-1})$$

and

$$h(1x_1x_2\cdots x_{n-1}) = 1h(x_1x_2\cdots x_{n-1}).$$

Then we have

$$h(S_x(y)) = h(x) \otimes h(y).$$

Since $j = h^{-1}$, we have

$$S_x(y) = j(h(x) \otimes h(y)).$$

This completes the proof.

Theorem 3.2 Let $x \in T$. Then the map $S_x|T: T \to T$ is a self-similarity map.

Proof Clearly, the map $S_x|T: T \to T$ is well defined. Let $T, U \in T$ with $T < U$, and let $\Gamma \in T$ with $S_x(T) < \Gamma < S_x(U)$. Then

$$a(h(S_x(T))) < a(h(\Gamma)) < a(h(S_x(U)))$$

By Lemma 3.1 we have

$$a(h(S_x(y))) = a(h(x)) * a(h(y)).$$

for $y \in \{T, U\}$. Then from the properties of $*$ -product, see pages 72 to 78 in reference [5] for more details, there exists a $W \in M$ with $a(h(T)) < W < a(h(U))$ such that

$$a(h(\Gamma)) = a(h(x)) * W.$$

Let $V = j(Z(a(W)))$. Then $T < V < U$ and

$$\Gamma = j(Z(a(h(x)) * a(h(V)))).$$

Thus by Lemma 3.1 one gets $S_x(V) = \Gamma$. This completes the proof.

Remark 3.3 Theorem 3.1 is useful to understand the local structure of the tree T. For example, for any even integer $n \geqslant 4$, the number of elements in n-level between the smallest one and $00101\cdots 0101\triangle$ ($= 0\triangle \otimes 0111\cdots 1\triangle$) is equal to the number of elements in $\frac{n}{2}$-level of T, this is easily deduced from Theorem 2.1 by taking $x = 0\triangle$.

4 Eigenvalues

In this section the eigenvalues of elements in T are defined, which are shown to be closely related to the spectral radiuses of the matrics induced by elements of T_{AR} (see reference [6] for more details). Also the elements of T are divided into two classes, the primary and the non-primary, which are shown to be closely related to the kneading sequences of tent maps.

Let $X(1) = 1, X(\triangle) = 0$ and $X(0) = -1$, For any $x = x_1x_2\cdots \in \sum$, define the formal power series (polynomial in fact, whenever $x \in T$)

$$d_x(t) = 1 + X(x_1)t + \sum_{i=2}^{|x|} t^i \prod_{j=1}^{i-1} X(x_j)\bar{X}(x_i),$$

where

$$\bar{X}(x_i) = \begin{cases} X(x'_i), & \text{if the parity of } 0's \text{ in } x_1x_2\cdots x_{i-1} \text{ is odd,} \\ X(x_i), & \text{if the parity of } 0's \text{ in } x_1x_2\cdots x_{i-1} \text{ is even.} \end{cases}$$

For any $a = a_1a_2\cdots \in \sum_{AR}$, define the formal power series (polynomial in fact, whenever $a \in T_{AR}$)

$$K_a(t) = 1 + \sum_{i=1}^{|a|} t^i \prod_{j=1}^{i} X(a_j).$$

Lemma 4.1 (i) Let $x \in \sum$ and $a \in \sum_{AR}$ with $a = h(x)$. Then $d_x(t) = K_a(t)$.

(ii)

$$D(t) = \begin{cases} d_x(t)\sum_{i=0}^{\infty} t^{ni}, & \text{if } x \in T, \\ d_x(t), & \text{if } x \in \sum - T, \end{cases}$$

where $D(t)$ is the related kneading determinant for the case $l = 2$ in reference [7].

(iii) Let $x \in T$ and $y \in \sum$. The $d_{S_x(y)}(t) = d_x(t)d_y(t^{|x|})$.

Proof (i) It suffices to show that, for any $2 \leqslant i \leqslant |x|$,

$$\prod_{j=1}^{i-1} X(x_j)\tilde{X}(x_i)\prod_{j=1}^{i} X(a_j) = 1.$$

Now we do it in following two cases

Case 1 $\prod_{j=1}^{i-1} X(x_j)\prod_{j=1}^{i-1} X(a_j) = 1$. We know, in this case, that the parity of $0'$s in $x_1x_2\cdots x_{i-1}$ is odd (even, resp.) if and only if the parity of $0'$s in $a_1a_2\cdots a_{i-1}$ is odd (even, resp). Note that the parity of $0'$s in $a_1a_2\cdots a_{i-1}$ is odd (even, resp.) if and only if $x_{i-1} = 0$ ($x_{i-1} = 1$, resp.). Then we have

$$\tilde{X}(x_i) = \begin{cases} X(x'_i), & \text{if the parity of } 0's \text{ in } x_1x_2\cdots x_{i-1} \text{ is odd,} \\ X(x_i), & \text{if the parity of } 0's \text{ in } x_1x_2\cdots x_{i-1} \text{ is even,} \end{cases}$$

$$= \begin{cases} X(x'_i), & \text{if the parity of } 0's \text{ in } a_1a_2\cdots a_{i-1} \text{ is odd,} \\ X(x_i), & \text{if the parity of } 0's \text{ in } a_1a_2\cdots a_{i-1} \text{ is even,} \end{cases}$$

$$= \begin{cases} X(x'_i), & \text{if } x_{i-1} = 0, \\ X(x_i), & \text{if } x_{i-1} = 1, \end{cases}$$

$$= X(a_i)$$

Thus the desired equality, in this case, is proved.

Case 2 $\prod_{j=1}^{i-1} X(x_j)\prod_{j=1}^{i-1} X(a_j) = -1$. A similar argument to case 1 can prove the desired equality.

(ii) Let $a = h(x)$. Then from Lemma 4.5 in reference [7] it is not difficult to obtain that

$$D(t) = \begin{cases} K_a(t)\sum_{i=0}^{\infty} t^{ni}, & \text{if } a \text{ is finite,} \\ K_a(t), & \text{if } a \text{ is infinite.} \end{cases}$$

Thus by (i) we have the desired equality.

(iii) Let $h(x) \equiv a = a_1a_2\cdots a_{n-1}\triangle$, $h(y) \equiv b = b_1b_2b_3\cdots$. Then by the proof of Lemma 3.1 we have $h(S_x(y)) = a \otimes b$. Put $T_i = \in (T_i), i = 1, 2, \cdots, n-1$, $V_j = \in (b_j)$ and $U_j = (-1)^{N(a)}V_j, j = 1, 2, \cdots |b| - 1$, where $N(a)$ is the parity of $0'$s in a. Clearly we have

$$T_1T_2\cdots T_{n-1}U_k = V_k, \text{ for any } 1 \leqslant k \leqslant |b|.$$

Note that

$$a \otimes b = a\hat{b}_1a\hat{b}_2a\hat{b}_3\cdots,$$

where

$$\hat{b}_i = \begin{cases} b'_i, & \text{if } N(a) \text{ is odd,} \\ b_i, & \text{if } N(a) \text{ is even.} \end{cases}$$

Then by (i) we have

$$\begin{aligned} d_{S_x(y)}(t) &= K_{a\otimes b}(t) \\ &= 1 + T_1t + \cdots + T_1T_2\cdots T_{n-1}t^{n-1} + \\ &\quad T_1T_2\cdots T_{n-1}U_1t^n + \cdots + \\ &\quad (T_1T_2\cdots T_{n-1})^2U_1U_2t^{2n} + \cdots \\ &= (1 + \sum_{i=1}^{n-1} T_1T_2 + \cdots + T_it^i)(1 + \\ &\quad \sum_{j=1}^{|b|} (T_1T_2\cdots T_{n-1})^j t^{jn} \prod_{k=1}^{j} U_k) \\ &= K_T(t)K_b(t^{|b|}) = d_x(t)d_y(t^{|x|}). \end{aligned}$$

This completes the proof.

From (ii) of Lemma 4.1 and Theorem 6.3 in reference [6] it follows that there exists an $s > 1$ such that $d_x(1/s) = 0$ and $d_x(t) \neq 0$ for any $0 < t < 1/s$ is $a(h(x)) > R^{*\infty}$ (see page 172 in reference [5]). Denote by d_x such an s and set $d_x = 1$ if $a(h(x)) \leqslant R^{*\infty}$

Definition 4.1 For an $x \in \sum$, the $d(x)$ is called to be the eigenvalue of x.

Definition 4.2 Let $x \in \sum$. Then x is called to be non-primary if there exist $y \in T - \{S_{0\triangle}^r(\triangle):$ $r \in Z_+\}$ and $z \in \sum - \{\triangle\}$ such that $x = S_y(z)$, where $S_{0\triangle}^r$ denotes the iterate of the map $S_{0\triangle}$, x is called to be primary if it is not non-primary.

Denote by P the set of primary sequences in $\sum$.

For $1 < \_ \leqslant 2$, let $f_{\_}(x) = \min\{\_x, \_(1-x)\}$ for any $x \in [0,1]$. Each $f_{\_}$ is so-called the tent map. Like in reference [5], denote by $K(f_{\_})$ the kneading sequence of $f_{\_}$.

Theorem 4.2 (i) $P = \{ j(Z(K(f_{\_}))) : 1 < \_ \leqslant 2\}$,

(ii) Let $x \in P$ and $x = j(Z(K(f_{\_})))$ for some $1 < \_ \leqslant 2$. The $d(x) = \_$.

(iii) Let $y \in \sum - P$. Then there exists uniquely an $x \in P$ such that $d(y) = d(x)$.

Proof (i) By Lemma 3.1 we know that an element $x \in \sum$ is primary if and only if $a(h(x))$ is primary in sense of reference [5]. Then by reference [5] and the monotonicity of kneading sequences of tent maps we have the desired conclusion.

(ii) Note that $f_{\_}$ has the topological entropy $h(f_{\_}) = \log \_$ for any $1 < \_ \leqslant 2$. Then, by reference [7] and (ii) of Lemma 4.1, we have $h(f_{\_}) = \log d(x) = \log \_$ and thus $d(x) = \_$.

(iii) Since $y \in \sum - P$, we that $a(h(y))$ is non-primary in the sense of reference [5]. Then there exist a finite primary sequence $P \in M - \{RC\}$ and a $Q \in M - \{C\}$ such that

$a(h(y)) = P * Q$

and such a P is unique. Put $x = j(Z(P))$ and $z = j(Z(Q))$. Then by Lemma 3.1 we have $y = S_x(z)$. Thus by (iii) of Lemma 4.1 and Definition 4.1 we have $d(y) = d(x)$. This completes the proof.

For any $1 \leqslant \lambda \leqslant 4$. let $g_\lambda(x) = \lambda x(1-x)$ for any $x \in [0,1]$. By Theorem Ⅲ 1.1 on page 173 in reference [5] we have $\sum = \{ j(Z(K(g_\lambda))) : 1 \leqslant \lambda \leqslant 4\}$ and then, For any $x \in T$, there exists a $\lambda \in [1,4]$ such that $K(g_\lambda) = a(h(x))$. Let $A_h(x)$ be the matrix induced by $h(x)$ in reference [1]. Then $h(g_\lambda) = \log d(A_h(x))$. On the other hand, by reference [7] and (ii) of Lemma 4.1 we have $h(g_\lambda) = \log d(x)$. Thus, by (iii) of Theorem 4.2 and Theorem of reference [1] we have

Proposition 4.3 $\overline{d(T)} \equiv \overline{\{d(x): x \in T\}} = \overline{\{d(x): x \in T \cap P\}} = [1,2]$.

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