DOI: 10.13656/j.cnki.gxkx.2000.02.007
广西科学 Guangxi Sciences 2000, 7(2): 103~ 104, 107

On the Order of Convergence for the Second Derivative of Hermite-Fejér Interpolation Polynomials

Hermite-Fejér插值多项式的二阶导数的收敛阶

Mu Lehua
木乐华

(Dept. of Math., Shandong Univ., Jinan, Shandong, 250100, China)
(山东大学数学系　山东济南　250100)

Abstract The estimate order of approximation for the second derivative of Hermite-Fejér interpolation polynomials based on the extended mixed Jacobi nodes is given.

Key words second derivative, Hermite-Fejér interpolation, extended mixed Jacobi nodes

摘要　给出扩充混合型 Jacobi节点的 Hermite-Fejér插值多项式的二阶导数的收敛阶.

关键词　二阶导数　Hermite-Fejér插值　扩充混合型 Jacobi节点
中图法分类号　O 174.21

Haverkamp, R.[1] and Muneer, Y. E.[2] have investigated the order of convergence for the higher derivatives of interpolation polynomials with respect to Chebyshev nodes. Here we discuss the order of convergence for the second derivative of Hermite-Fejér interpolation polynomials with respect to the extended mixed Jacobi nodes.

Let $w(x) = (1-x)\dfrac{\sin\frac{2n+1}{2}\theta}{\sin\frac{\theta}{2}}$ $(x = \cos\theta)$, (1)

and its zeros (the extended mixed Jacobi nodes) are $\{x_k\}_0^n$:

$$x_0 = 1,\quad x_k = \cos\theta_k,$$
$$\theta_k = \frac{2k\pi}{2n+1}\quad (k = 1,2,\cdots,n). \tag{2}$$

For these nodes the Lagrange fundamental polynomials[3] are

$$l_k(x) = a_k\frac{w(x)}{x-x_k}\quad \left(a_k = \frac{1}{w'(x_k)}\right).$$

By (1) and

$$w'(x) = -\left(\frac{\sin\frac{2n+1}{2}\theta}{2\sin\frac{\theta}{2}} + \frac{(2n+1)\cos\frac{2n+1}{2}\theta}{2\cos\frac{\theta}{2}}\right),$$
$$x = \cos\theta, \tag{3}$$

passing direct calculation, we get

$$a_0 = -\frac{1}{2n+1},\ a_k = \frac{(-1)^{k+1}2\cos\frac{\theta_k}{2}}{2n+1}, \tag{4}$$

and

$$l_0(x) = \frac{\sin\frac{2n+1}{2}\theta}{(2n+1)\sin\frac{\theta}{2}},$$
$$l_k(x) = \frac{(-1)^{k+1}4\sin\frac{\theta}{2}\cos\frac{\theta_k}{2}\sin\frac{2n+1}{2}\theta}{(2n+1)(x-x_k)}. \tag{5}$$

Let $f(x)\in C^2[-1,1]$. For these nodes its Hermite-Fejér interpolation polynomials $H_{2n+1}(f;x)$ which satisfy

$$H_{2n+1}(f;x_k) = f(x_k),\ H'_{2n+1}(f;x_k) = 0,$$
$$k = 0,1,\cdots,n$$

are

$$H_{2n+1}(f;x) = \sum_{k=0}^{n} f(x_k)V_k(x)l_k^2(x), \tag{6}$$

where

$$V_k(x) = 1-(x-x_k)\frac{k''(x_k)}{k'(x_k)}$$
$$= \begin{cases} 1+\frac{2}{3}n(n+1)(1-x),\ k=0, \\ 1+\frac{x-x_k}{\sin^2\theta_k},\ k = 1,2,\cdots,n. \end{cases} \tag{7}$$

$\|\cdot\|$ denotes the supremum norm on $[-1, 1]$ and write

$$f_{x,0}(x_k) = \frac{f(x_k)-f(x)}{x_k-x},$$
$$f_{x,1}(x_k) = \frac{f_{x,0}(x_k)-f'(x)}{x_k-x},$$

1999-12-16收稿.

$k = 0, 1, \cdots, n$ (8)

Lemma 1 If $f(x) \in C^2[-1,1]$, then for $x \in (-1, 1)$,

$$H''_{2n+1}(f; x) = \frac{O(n^2)}{1 - x^2} \| f'' \| ,$$

hereafter the bounds of the terms "O" are absolute constants.

Proof We may suppose without loss of generality that $f'(-1) = 0$. (9)

From a known property[3] $\sum_{k=0}^{n} V_k(x) l_k^2(x) = 1$ we get $\sum_{k=0}^{n} V_k(x) l_k^2(x)^{(i)} = 0, i = 1, 2,$ so by (6),

$$H''_{2n+1}(f; x) = \sum_{k=0}^{n} (f(x_k) - f(x)) \cdot$$

$$(V_k(x) l_k^2(x))'' - 3\sum_{k=0}^{n} f'(x)(V_k(x) l_k^2(x))'. \quad (10)$$

Noticing that

$$(x_k - x) l_k'(x) = l_k(x) - a_k \omega'(x),$$
$$(x_k - x) l_k''(x) = 2l_k'(x) - a_k \omega''(x), \quad (11)$$

and using the notations in (8), the above equality (10) can be written in the form as follows

$$H''_{2n+1}(f; x) = 4\sum_{k=0}^{n} f_{x,0}(x_k) V_k'(x) l_k(x)(l_k(x) - a_k \omega'(x)) + 6\sum_{k=0}^{n} f_{x,1}(x_k) V_k(x) l_k(x)(l_k(x) - a_k \omega'(x)) - 2\sum_{k=0}^{n} f_{x,0}(x_k) V_k(x) l_k'(x) a_k \omega'(x) -$$
$$2\sum_{k=0}^{n} f_{x,0}(x_k) V_k(x) l_k(x) a_k \omega''(x) - 3\sum_{k=0}^{n} f'(x) V_k'(x) l_k^2(x) \stackrel{\text{def}}{=} \sum_{i=0}^{5} I_i. \quad (12)$$

First estimate I_1.

From (3) ~ (5), it follows that

$$\omega'(x) = \frac{O(n)}{\sqrt{1+x}}, a_k = O(\frac{1}{n}), l_k(x) = O(1),$$
$$x \in (-1, 1), k = 0, 1, \cdots, n, \quad (13)$$

so

$$l_k(x) - a_k \omega'(x) = \frac{O(1)}{\sqrt{1+x}}, x \in (-1, 1). \quad (14)$$

By (8) using Lagrange mean theorem we have

$$|f_{x,0}(x_k)| \leqslant \| f' \| , |f_{x,1}(x_k)| \leqslant \| f'' \| . \quad (15)$$

Combining (13) ~ (15), it follows that

$$\sum_{k=1}^{n} f_{x,0}(x_k) V_k'(x) l_k(x)(l_k(x) - a_k \omega'(x))$$
$$= \frac{O(1)}{\sqrt{1+x}} \| f' \| \sum_{k=1}^{n} | V_k'(x) | ,$$

but by (7) and $\theta_k = \frac{2k c}{2n+1}, k = 1, 2, \cdots, n,$

$$\sum_{k=1}^{n} | V_k'(x) | = \sum_{k=1}^{n} \frac{1}{\sin^2\theta_k} = O(n^2), \quad (16)$$

and we also have

$$f_{x,0}(1) V_0'(x) l_0(x)(l_0(x) - a_0 \omega'(x))$$
$$= \frac{O(n^2)}{\sqrt{1 - x^2}} \| f' \| ,$$

thus we obtain the estimate $I_1 = \frac{O(n^2)}{\sqrt{1 - x^2}} \| f' \|$, $x \in (-1, 1)$.

Similarly, we have $I_2 = \frac{O(n^2)}{\sqrt{1 - x^2}} \|f''\|$, $x \in (-1, 1)$.

Next we estimate I_3.

From (7) and (11), we get

$$\sum_{k=1}^{n} f_{x,0}(x_k) V_k(x) l_k'(x) a_k \omega'(x)$$
$$= \sum_{k=1}^{n} f_{x,0}(x_k) l_k'(x) a_k \omega'(x) + \sum_{k=1}^{n} f_{x,0}(x_k) \frac{l_k(x) - a_k \omega'(x)}{\sin^2\theta_k} \cdot a_k \omega'(x).$$

According to Markov's inequality and (13),

$$l_k'(x) = \frac{O(n)}{\sqrt{1 - x^2}}, x \in (-1, 1).$$

Again by (13) ~ (16), we can conclude that

$$\sum_{k=1}^{n} f_{x,0}(x_k) V_k(x) l_k'(x) a_k \omega'(x)$$
$$= O(1)\|f'\| \left\{ \sum_{k=1}^{n} | l_k'(x) a_k \omega'(x) | + \sum_{k=1}^{n} \frac{|(l_k'(x) - a_k \omega'(x)) a_k \omega'(x)|}{\sin^2\theta_k} \right\}$$
$$= \frac{O(n^2)}{(1+x)\sqrt{1 - x}} \|f'\|.$$

Moreover,

$$f_{x,0}(1) V_0(x) l_0'(x) a_0 \omega'(x)$$
$$= \frac{O(n^2)}{(1+x)\sqrt{1 - x}} \|f'\| , \text{ so}$$

$$I_3 = \frac{O(n^2)}{(1+x)\sqrt{1 - x}} \|f'\|, x \in (-1, 1).$$

Finally, we estimate I_4 and I_5.

With help of the formula (3) we can conclude that

$$\omega''(x) = \frac{O(n^2)}{1 - x^2}, x \in (-1, 1).$$

Again by (7) and (13), (15), (16), it follows that

$$\sum_{k=1}^{n} f_{x,0}(x_k) V_k(x) l_k(x) a_k \omega''(x) = \frac{O(n^2)}{\sqrt{1 - x^2}} \|f'\| + \sum_{k=1}^{n} f_{x,0}(x_k) \frac{x - x_k}{\sin^2\theta_k} l_k(x) a_k \omega''(x).$$

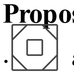
but by (5) we know that $(x - x_k) l_k(x) = O(\frac{1}{n})$, so

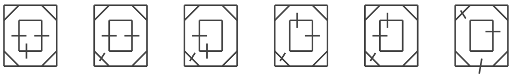
$$I_4 = \frac{O(n^2)}{\sqrt{1 - x^2}} \|f'\|, x \in (-1, 1).$$

For I_5, we have $I_5 = -3\sum_{k=0}^{n} f'(x) V_k'(x) l_k^2(x) = O(n^2)\|f'\|$.

(下转第 107页 Continue on page 107)

Due to "020" ≈ "200", and the others are not homeomorphic with each other, the proposition has been proved.

Proposition The homeomorphism class of G. M.  are six as follows:

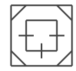
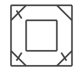
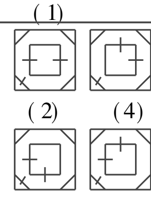
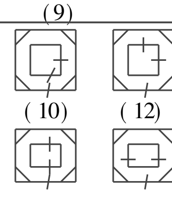
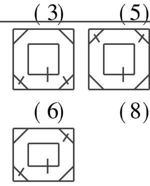
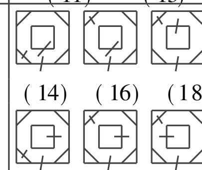


Proof By theorem, finding homeomorphism class is equal to homeomorphism class which has three negative edges. Let negative edge be outer, radiative and inner, and their numbers are respectively a, b and c, and $a, b, c \in \{0, 1, 2, 3\}$, $a+b+c=3$. Showed in Table 2.

Through twisting, it can be easy to see that: (6) ≈ (1), (7) ≈ (1), (8) ≈ (1), (9) ≈ (2), (11) ≈ (10), (12) ≈ (1), (13) ≈ (1), (16) ≈ (14), (18) ≈ (14), (19) ≈ (15), and due to symmetry, (10) ≈ (2), (15) ≈ (4), (14) ≈ (1). So their homeomorphism classes are exactly six: (1), (2), (3), (4), (5) and (17).

The proposition has been proved.

Table 2

a b c	Deputy class	a b c	Deputy class
003	 (1)	030	 (9)
012	 (2) (4) (3) (5)	102	 (10) (12) (11) (13)
021	 (6) (8) (7)	111	 (14) (16) (18) (15) (17) (19)

Reference

1 Liu Y X, Li Q S. Graphlike manifolds. Chinese Quarterly Journal of Math. 1994, 9 (4): 46~ 51.

(责任编辑: 蒋汉明)

(上接第104页 Continue from page 104)

From (9), we have $|f'(x)| = \left|\int_{-1}^{x} f''(t)\,dt\right| \leqslant 2\|f''\|$, $x \in (-1, 1)$.

To sum up we get from (13) the conclusion of Lemma 1.

Let $f(x) \in C^2[-1, 1]$ and $P(x)$ is a polynomial such that $P''(x)$ is the best approximation polynomial of degree n of $f''(x)$. From $H''_{2n+1}(P; x) = P''(x)$[3], we have

$H''_{2n+1}(f; x) - f''(x) = H''_{2n+1}(f - P; x) + P''(x) - f''(x)$,

further applying Lemma 1 we immediately obtain the following Theorem 1.

Theorem 1 If $f(x) \in C^2[-1, 1]$, then for $x \in (-1, 1)$ we have

$$H''_{2n+1}(f; x) - f''(x) = \frac{O(n^2)}{1-x^2} E_n(f''),$$

where $E_n(f'')$ is the best approximation of f'' by polynomials of degree n.

Theorem 2 If $f(x) \in C^p[-1, 1]$ and $\omega(f^{(p)}; \delta)$ is the modulus of continuity of $f^{(p)}$, then for $x \in (-1, 1)$, we have

$$H''_{2n+1}(f; x) - f''(x) = O\left(\frac{1}{n^{p-4}}\right) \frac{1}{1-x^2} \omega\left(f^{(p)}; \frac{1}{n}\right), \quad p \geqslant 4.$$

Proof Using Theorem 1 and the Jackson theorem:

$$E_n(f'') = O\left(\frac{1}{n^{p-2}}\right) \omega\left(f^{(p)}; \frac{1}{n}\right),$$

we obtain immediately the conclusion of Theorem 2.

Similarly, by $\bar{H}_{2n+1}(f; x)$ denote Hermite-Fejér interpolation polynomials based on the zeros of $\omega(x) = (1+x)\dfrac{\cos\frac{2n+1}{2}\theta}{\cos\frac{\theta}{2}}$, $x = \cos\theta$ (the other mixed Jacobi nodes) we also can conclude the following theorem.

If $f(x) \in C^p[-1, 1]$, then for $x \in (-1, 1)$ we have

$$\bar{H}''_{2n+1}(f; x) - f''(x) = \frac{O(n^2)}{1-x^2} E_n(f'') = O\left(\frac{1}{n^{p-4}}\right) \frac{1}{1-x^2} \omega\left(f^{(p)}; \frac{1}{n}\right), \quad p \geqslant 4.$$

References

1 Haverkamp R. Approximationseigenschaften differenzierter interpolationspolynome, J Approx Theory, 1978, 23: 261 ~ 266.

2 Muneer Y E. On lagrange and hermite interpolation 1, Acta Math Hung, 1987, 49 (3- 4): 293~ 305.

3 Natanson I P. Constructive Function Theory, 1965.

(责任编辑: 蒋汉明)