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Generalization of Meyer Wavelet
Meyer小波的推广

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Abstract The generalized Meyer wavelet is discussed.

Key words Meyer wavelet, scaling function, system transfer function, multiresolution analysis

摘要 讨论 Meyer小波的推广。

关键词 Meyer小波 尺度函数 传递函数 多尺度分析
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1 Introduction and main result

Denote the Fourier transform of f by $\hat{f}$, and the compact support of f by supp f, that is

$$\text{supp} f = \text{clos}\{t, f(t) \neq 0\}.$$

If $j \in L^2(-\infty, \infty)$ and the system of functions

$$\{2^{\frac{m}{2}} j(2^m t - n), m, n \in Z\}$$

is an orthonormal basis, then we call j the mother wavelet[1,2].

The construction of mother wavelet is based on a multiresolution analysis.

If a sequence of closed subspaces $\{V_m\}_{m\in Z}$ in $L^2(-\infty, \infty)$ satisfies the following conditions

(i) $V_m \subset V_{m+1} (m \in Z), \overline{\bigcup_{m\in Z} V_m} = L^2(-\infty, \infty)$, $\bigcap_{m\in Z} V_m = \{0\}$,

(ii) $f(t) \in V_m \leftrightarrow f(2t) \in V_{m+1}, m \in Z$,

(iii) there is a $h(t) \in V_0$ such that $\{h(t-n)\}_{n\in Z}$ is an orthonormal basis of V_0.

Then we call $\{V_m\}_{m\in Z}$ a multiresolution analysis, and $h(t)$ a scaling function[1,2].

Theorem A[3] Let a, b satisfy the following inequalities

$$a < 0, b > 0, \frac{b}{2} - a \leq 2c, b - \frac{a}{2} \leq 2c, b - a > 2c \quad (1)$$

and let the real-valued continuous function $\hat{h}(k)$ satisfy

(i) supp $\hat{h}(k) = [a, b]$.

(ii) $|\hat{h}(k)| = 1, a + T \leq k \leq b - T$ $(T = b - a - 2c)$.

(iii) $|\hat{h}(k)|^2 + |\hat{h}(k + 2c)|^2 = 1, a \leq k \leq a + T$.

Then its inverse Fourier transform $h(t)$ is a scaling function.

Theorem B[2] Let $h(t)$ be a scale function, then there is a system transfer function $H(k) \in L^2(-c, c)$ which is a $2c$-periodic function such that

$$\hat{h}(k) = H(\frac{k}{2})\hat{h}(\frac{k}{2}).$$

Theorem C[2] Let $h(t)$ be a scaling function, and $H(k)$ be a corresponding system transfer function. Then the corresponding mother wavelet $j(t)$ satisfies the following formula

$$\hat{j}(k) = e^{-i\frac{k}{2}}\hat{h}(\frac{k}{2})\overline{H(\frac{k}{2} + c)}.$$

In 1986[4] Meyer constructed the first mother wavelet $j(t)$ (so-called Meyer wavelet). Its explicit expression is stated as follows.

$$e^{i\frac{k}{2}}\hat{j}(k) = \begin{cases} \sin[\frac{c}{2}\nu(\frac{3}{2c}|k| - 1)], & \frac{2}{3}c \leq |k| \leq \frac{4}{3}c, \\ \cos[\frac{c}{2}\nu(\frac{3}{4c}|k| - 1)], & \frac{4}{3}c \leq |k| \leq \frac{8}{3}c, \\ 0, & \text{otherwise}, \end{cases} \quad (2)$$

where $\nu(x)$ is differentiable infinitely often and

$$\nu(x) = \begin{cases} 0, x \leq 0, \\ 1, x \geq 1, \end{cases} \quad (3)$$

$$\nu(x) + \nu(1 - x) = 1, (-\infty < x < \infty). \quad (4)$$

Now we give a generalization of the above result.

Theorem Let a, b satisfy Inequalities (1), and let $\nu_{a,b}(x)$ be differentiable infinitely often and

$$\nu_{a,b}(x) = \begin{cases} 0, x \leq 0, \\ 1, x \geq Z, \end{cases} \quad (5)$$

and $\nu_{a,b}(x) + \nu_{a,b}(Z - x) = 1 \ (-\infty < x < \infty)$, (6)

here $Z = -\frac{2(b - a) - 4c}{b - a - 4c}$. (7)

Then the function $j_{a,b}(t)$ defined by the following for-

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mula is a mother wavelet

$$e^{i\frac{\omega}{2}}\hat{\psi}_{a,b}(\omega)=\begin{cases}\sin[\frac{\pi}{2}\nu_{a,b}(\frac{1}{2\pi-\frac{b-a}{2}}|\omega-\frac{a+b}{2}|-1)], & (2\pi-\frac{b-a}{2}\leqslant|\omega-\frac{a+b}{2}|\leqslant\frac{b-a}{2}),\\ 1, & (2b-4\pi\leqslant\omega\leqslant a\ \text{or}\ b\leqslant\omega\leqslant 2a+4\pi),\\ \cos[\frac{\pi}{2}\nu_{a,b}(\frac{1}{4\pi-(b-a)}|\omega-(a+b)|-1)], & (4\pi+(a-b)\leqslant|\omega-(a+b)|\leqslant b-a),\\ 0, & \text{otherwise}.\end{cases}\qquad(8)$$

Specially, for $a=-\frac{4}{3}\pi, b=\frac{4}{3}\pi$, this $\psi_{a,b}(t)$ is just Meyer wavelet.

2 Lemmas

Lemma 1 Let a, b and $\nu_{a,b}(x)$ are stated as Theorem. Then the function $\varphi_{a,b}(t)$ defined by the following formula

$$\hat{\varphi}_{a,b}(\omega)=\cos[\frac{\pi}{2}\nu_{a,b}(\frac{1}{2\pi-\frac{b-a}{2}}|\omega-\frac{a+b}{2}|-1)]\qquad(9)$$

is a scaling function.

Proof From (5), (7) and (9), it follows that

$$\hat{\varphi}_{a,b}(\omega)=\begin{cases}1, & |\omega-\frac{a+b}{2}|\leqslant 2\pi-\frac{b-a}{2},\\ 0, & |\omega-\frac{a+b}{2}|\geqslant\frac{b-a}{2}.\end{cases}$$

Let $\delta=b-a-2\pi$, we have

$$\hat{\varphi}_{a,b}(\omega)=1,\ \omega\in[a+\delta, b-\delta]\qquad(10)$$

and $$\text{supp}\,\hat{\varphi}_{a,b}(\omega)=[a,b].\qquad(11)$$

Next we prove

$$|\hat{\varphi}_{a,b}(\omega)|^2+|\hat{\varphi}_{a,b}(\omega+2\pi)|^2=1,\ \omega\in[a,a+\delta].\qquad(12)$$

In fact, for $a\leqslant\omega\leqslant a+\delta$, we obtain from $b-\frac{a}{2}\leqslant 2\pi, b>0, \delta=b-a-2\pi$ that

$$\omega\leqslant a+\delta=b-2\pi\leqslant\frac{a}{2}\leqslant\frac{a+b}{2}.$$

So by (9), we have

$$\hat{\varphi}_{a,b}(\omega)=\cos\left[\frac{\pi}{2}\nu_{a,b}\left(\frac{b-\omega-2\pi}{2\pi-\frac{b-a}{2}}\right)\right].\qquad(13)$$

On the other hand, for $a\leqslant\omega\leqslant a+\delta$, we obtain from $\frac{b}{2}-a\leqslant 2\pi, a<0$ that

$$\omega+2\pi\geqslant a+2\pi\geqslant\frac{b}{2}\geqslant\frac{b+a}{2}.$$

So by (9) and (7), we have

$$\hat{\varphi}_{a,b}(\omega+2\pi)=\cos\left[\frac{\pi}{2}\nu_{a,b}\left(\frac{\omega-a}{2\pi-\frac{b-a}{2}}\right)\right]=\cos\left[\frac{\pi}{2}\nu_{a,b}\left(1-\frac{b-\omega-2\pi}{2\pi-\frac{b-a}{2}}\right)\right].$$

Again applying Formula (6), we have

$$\hat{\varphi}_{a,b}(\omega+2\pi)=\cos\left[\frac{\pi}{2}\left(1-\nu_{a,b}\left(\frac{b-\omega-2\pi}{2\pi-\frac{b-a}{2}}\right)\right)\right]=\sin\left[\frac{\pi}{2}\nu_{a,b}\left(\frac{b-\omega-2\pi}{2\pi-\frac{b-a}{2}}\right)\right].\qquad(14)$$

Now from (13) and (14), we know that (12) is valid.

By (10)~(12), applying Theorem A, we conclude Lemma 1.

Lemma 2 The system transfer function corresponding to the scaling function $\varphi_{a,b}(t)$ is

$$H_{a,b}(\omega)=\sum_{k\in Z}\hat{\varphi}(2\omega+4k\pi),\ \omega\in(-\infty,\infty).\qquad(15)$$

Proof From supp $\hat{\varphi}(\omega)=[a,b]$, we know that the above summation formula has at most finite terms.

We first prove that

$$\hat{\varphi}_{a,b}(\omega)=\hat{\varphi}_{a,b}(\omega)\hat{\varphi}_{a,b}(\frac{\omega}{2}),\ \omega\in(-\infty,\infty).\qquad(16)$$

In fact, for $\omega\in[a,b]$, $\frac{\omega}{2}\in[\frac{a}{2},\frac{b}{2}]$, from $\frac{b}{2}-a\leqslant 2\pi, b-a=2\pi+\delta$, we know that $\frac{b}{2}\geqslant\delta$, i.e. $\frac{b}{2}\leqslant b-\delta$. Similarly we have $\frac{a}{2}\geqslant a+\delta$, so

$$[\frac{a}{2},\frac{b}{2}]\subset[a+\delta,b-\delta].$$

Again by (10), $\hat{\varphi}_{a,b}(\frac{\omega}{2})=1, \omega\in[a,b]$, but by (11), $\hat{\varphi}_{a,b}(\omega)=0, \omega\notin[a,b]$, so (16) is valid.

In view of supp $\hat{\varphi}_{a,b}(\omega+4k\pi)=[a+4k\pi, b+4k\pi]$, supp $\hat{\varphi}_{a,b}(\frac{\omega}{2})=[2a,2b]$ and inequalities $2b\leqslant a+4\pi, b-4\pi\leqslant 2a$, we get that for $k\neq 0$,

$$\text{supp}\,\hat{\varphi}_{a,b}(\omega+4k\pi)\cap\text{supp}\,\hat{\varphi}_{a,b}(\frac{\omega}{2})=\varnothing.$$

From this and (16), it follows that

$$\hat{\varphi}_{a,b}(\omega)=(\sum_{k}\hat{\varphi}_{a,b}(\omega+4k\pi))\hat{\varphi}_{a,b}(\frac{\omega}{2}),\ \omega\in(-\infty,\infty).\qquad(17)$$

But by Theorem B, we know that the system transfer function $H_{a,b}(\omega)$ satisfies

$$\hat{\varphi}_{a,b}(\omega)=H_{a,b}(\frac{\omega}{2})\hat{\varphi}_{a,b}(\frac{\omega}{2}),\ \omega\in(-\infty,\infty).\qquad(18)$$

Since supp $\hat{\varphi}_{a,b}(\frac{\omega}{2})=[2a,2b]$, comparing (17) with (18), we obtain that

$$H_{a,b}(\frac{\omega}{2})=\sum_{k}\hat{\varphi}_{a,b}(\omega+4k\pi)\quad\omega\in[2a,2b].$$

Again noticing that the functions of both sides of the above formula are all 4π-periodic functions and $2b-2a>4\pi$, hence in the whole real axis, (18) is valid. Lemma 2 is proved.

3 Proof of Theorem

From Theorem C and Lemma 2, we know the

mother wavelet $\psi_{a,b}(t)$ corresponding to the scaling function $\varphi_{a,b}(t)$ satisfies

$$\hat{\psi}_{a,b}(\omega) = e^{-i\frac{\omega}{2}}\sum_k \hat{\varphi}_{a,b}(\omega+(4k+2)\pi)\hat{\varphi}_{a,b}(\frac{\omega}{2}),$$

$\omega\in(-\infty,\infty)$. (19)

By Inequalities 1 and $\text{supp}\,\hat{\varphi}_{a,b}(\omega)=[a,b]$, we obtain that

$$\text{supp}\,\hat{\varphi}_{a,b}(\omega+(4k+2)\pi)\cap\text{supp}\,\hat{\varphi}_{a,b}(\frac{\omega}{2})=\varnothing,$$

$(k\neq 0,-1)$.

So from (19), we have

$$\hat{\varphi}_{a,b}(\omega) = e^{-i\frac{\omega}{2}}(\hat{\varphi}_{a,b}(\omega+2\pi)+\hat{\psi}_{a,b}(\omega-2\pi))\hat{\varphi}_{a,b}(\frac{\omega}{2}),\ \omega\in(-\infty,\infty). \quad (20)$$

From $\text{supp}\,\hat{\varphi}_{a,b}(\omega)=[a,b]$, we know that

$$\text{supp}\{\hat{\varphi}_{a,b}(\omega+2\pi)+\hat{\varphi}_{a,b}(\omega-2\pi)\}=[a-2\pi,b-2\pi]\cup[a+2\pi,b+2\pi] \quad (21)$$

and

$$\text{supp}\,\hat{\varphi}_{a,b}(\frac{\omega}{2})=[2a,2b]. \quad (22)$$

Again from (1), we have

$$2a\in(a-2\pi,b-2\pi),\ 2b\in(a+2\pi,b+2\pi). \quad (23)$$

By (21)~(23) we conclude that

$$\text{supp}\,\hat{\psi}_{a,b}(\omega)=[2a,b-2\pi]\cup[a+2\pi,2b]. \quad (24)$$

Next, we prove that Formula (18) is valid.

We divide the $\text{supp}\,\hat{\psi}_{a,b}(\omega)$ into six intervals, that is $[2a,b-2\pi]=[2a,2b-4\pi]+[2b-4\pi,a]+[a,b-2\pi]$ and

$[a+2\pi,2b]=[a+2\pi,b]+[b,2a+4\pi]+[2a+4\pi,2b]$.

(i) Let $\omega\in[2a,2b-4\pi]$.

Now by (1) and $\delta=b-a-2\pi$, we have

$$\omega+2\pi\geqslant 2a+2\pi\geqslant b-2\pi=a+\delta$$

and

$$\omega+2\pi\leqslant 2b-2\pi\leqslant a+2\pi=b-\delta.$$

So from (10) it follows that

$$\hat{\varphi}_{a,b}(\omega+2\pi)=1.$$

Now $\omega-2\pi\leqslant 2b-6\pi\leqslant a$, so by (11), we have

$$\hat{\varphi}_{a,b}(\omega-2\pi)=0.$$

From this and (20), (9), we get

$$e^{i\frac{\omega}{2}}\hat{\psi}_{a,b}(\omega)=\hat{\varphi}_{a,b}(\frac{\omega}{2})=\cos[\frac{\pi}{2}\nu_{a,b}(\frac{1}{2\pi-\frac{b-a}{2}}|\frac{\omega}{2}-\frac{a+b}{2}|-1)].$$

(ii) Let $\omega\in[2b-4\pi,a]$.

Now $a+\delta\leqslant\omega+2\pi\leqslant a+2\pi=b-a$, so $\hat{\varphi}_{a,b}(\omega+2\pi)=1$.

Again by $\omega-2\pi\leqslant a-2\pi<a$, we have $\hat{\varphi}_{a,b}(\omega-2\pi)=0$. Since $\frac{\omega}{2}\geqslant b-2\pi=a+\delta$, $\frac{\omega}{2}\leqslant\frac{a}{2}<0<b-\delta$, we have $\hat{\varphi}_{a,b}(\frac{\omega}{2})=1$.

Hence $\hat{\psi}_{a,b}(\omega)=1$, $\omega\in[2b-4\pi,a]$.

(iii) Let $\omega\in[a,b-2\pi]$.

Now $\omega-2\pi\leqslant b-4\pi$, but by (1), $b-a\leqslant\frac{4}{3}\pi$, so $\omega-2\pi<a$, further $\hat{\varphi}_{a,b}(\omega-2\pi)=0$.

On the other hand, $\frac{\omega}{2}\geqslant\frac{a}{2}\geqslant b-2\pi=a+\delta$ and $\frac{\omega}{2}\leqslant\frac{b-2\pi}{2}\leqslant a+\pi<b-\delta$, so

$$\hat{\varphi}_{a,b}(\frac{\omega}{2})=1.$$

Hence from (20) and $\omega+2\pi\geqslant\frac{b-a}{2}$,

$$e^{i\frac{\omega}{2}}\hat{\psi}_{a,b}(\omega)=\hat{\varphi}_{a,b}(\omega+2\pi)=\cos[\frac{\pi}{2}\nu_{a,b}(\frac{1}{2\pi-\frac{b-a}{2}}(\omega+2\pi-\frac{b-a}{2})-1)].$$

Similarly we have

(iv) For $\omega\in[a+2\pi,b]$,

$$e^{i\frac{\omega}{2}}\hat{\psi}_{a,b}(\omega)=\cos[\frac{\pi}{2}\nu_{a,b}(\frac{1}{2\pi-\frac{b-a}{2}}(\frac{a+b}{2}-\omega+2\pi)-1)].$$

(v) For $\omega\in[b,2a+4\pi]$, $e^{i\frac{\omega}{2}}\hat{\psi}_{a,b}(\omega)=1$.

(vi) For $\omega\in[2a+4\pi,2b]$,

$$e^{i\frac{\omega}{2}}\hat{\psi}(\omega)=\cos[\frac{\pi}{2}\nu_{a,b}(\frac{1}{2\pi-\frac{b-a}{2}}|\frac{\omega}{2}-\frac{a+b}{2}|-1)].$$

Combining (iii) with (iv), we have

$$e^{i\frac{\omega}{2}}\hat{\psi}_{a,b}(\omega)=\cos[\frac{\pi}{2}\nu_{a,b}(\frac{1}{2\pi-\frac{b-a}{2}}(2\pi-|\omega-\frac{a+b}{2}|)-1)],$$

$$(2\pi-\frac{b-a}{2}\leqslant|\omega-\frac{a+b}{2}|\leqslant\frac{b-a}{2}). \quad (25)$$

Again combining (i) with (vi), we get

$$e^{i\frac{\omega}{2}}\hat{\psi}_{a,b}(\omega)=\cos[\frac{\pi}{2}\nu_{a,b}(\frac{1}{4\pi-(b-a)}|\omega-(a+b)|-1)],\ (4\pi+a-b\leqslant|\omega-(a+b)|\leqslant b-a). \quad (26)$$

By (ii) and (v),

$$e^{i\frac{\omega}{2}}\hat{\psi}_{a,b}(\omega)=1\quad \omega\in(2b-4\pi,a)\cup(b,2a+4\pi). \quad (27)$$

Finally, from (25)~(27) and (24), we obtain the Formula (8). Theorem is proved.

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