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One Property of the Singular Values of Matrix
关于矩阵奇异值的一个性质

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Abstract　A new result about the relation between the singular values of an arbitrary matrix and those of its submatrix, which is called generalized interlacing theorem, is presented.

Key words　matrix, submatrix, singular value, interlacing theorem

摘要　给出关于任意一矩阵与子矩阵的奇异值关系，称之为广义分隔定理，并给出其一个简单应用。

关键词　矩阵　子矩阵　奇异值　分隔定理
中图法分类号　O 151.21

As to the singular values, there is interlacing theorem in Reference [1]. And it comes from Cauchy interlacing theorem:

Let $A \in C^{m \times n}$ be a given matrix and $A1$ be the matrix obtained by deleting any one column from A. Let $\{\sigma_i\}$ denote the singular values of A and $\{\theta_i\}$ denote the singular values of $A1$, both arrange in non-increasing order.

(i) If $m \geqslant n$, then

$$\sigma_1 \geqslant \theta_1 \geqslant \sigma_2 \geqslant \theta_2 \geqslant \cdots \geqslant \theta_{n-1} \geqslant \sigma_n \geqslant 0.$$

(ii) If $m < n$, then

$$\sigma_1 \geqslant \theta_1 \geqslant \sigma_2 \geqslant \theta_2 \geqslant \cdots \geqslant \sigma_m \geqslant \theta_m \geqslant 0.$$

If a row of A is deleted and replaced with a column, the inequalities associated with the two cases (i) and (ii) are interchanged.

But what happen to the matrix A and its submatrix $A1$ which is obtained by deleting any one row and the same column? We try to work out this question in this paper, and an application about some normal matrixes is obtained.

1　Results and Application

Firstly, let us introduce the result of Reference [1] on Cauchy interlacing theorem.

Lemma 1 (Cauchy interlacing theorem)　Let A be a Hermitian matrix, $A1$ be the submatrix of A which is obtained by deleting some row and the same column from A. $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A in increasing order. $\mu_1, \mu_2, \cdots, \mu_{n-1}$ are the eigenvalues of $A1$ in increasing order, then

$$\lambda_1 \leqslant \mu_1 \leqslant \cdots \leqslant \lambda_{n-1} \leqslant \mu_{n-1} \leqslant \lambda_n.$$

Theorem 1　Let $A \in R^{n \times n}$ be Hermitian and $z \in R^n$ be a given vector. If the eigenvalues of A and $A + zz^T$ are arranged in increasing order

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n,$$

We have

$$\lambda_1(A) \leqslant \lambda_1(A + zz^T) \leqslant \cdots \leqslant \lambda_n(A) \leqslant \lambda_n(A + zz^T).$$

Proof　Since zz^T is a positive semidefinite matrix, we obtain $\lambda_i(A) \leqslant \lambda_i(A + zz^T)$ and $\lambda_i(A + zz^T) \leqslant \lambda_{i+1}(A)$ (see in Reference [1]).

We show our results as follows.

Theorem 2　If $B \in R^{k \times k}$ is a matrix, $\{\sigma_i\}_{i=1}^{k}$ are the singular values of B and $\{\theta_i\}_{i=1}^{k-1}$ are the singular values of $B1 = B(1, \cdots, i-1, i+1, \cdots, k)$ which is a matrix by deletion of the ith row and the ith column of B, and both are arranged in increasing order, then there exists the sequence of nonnegative real numbers $\{K_i\}_{i=1}^{k-1}$ which satisfies

$$\sigma_1 \leqslant K_1 \leqslant \sigma_2 \leqslant \cdots \leqslant \sigma_{k-1} \leqslant K_{k-1} \leqslant \sigma_k,$$

meanwhile,

$$\theta_1 \leqslant K_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_{k-1} \leqslant K_{k-1}.$$

Proof　Denote the singular values of B

$$\sigma_1, \sigma_2, \cdots, \sigma_k \text{ where } \sigma_i \geqslant 0, i = 1, 2, \cdots, k,$$

and $\{\sigma_i\}$ are arranged in increasing order.

That is, the eigenvalues of Hermitian matrix BB^* in increasing order are

$$\sigma_1^2, \sigma_2^2, \cdots, \sigma_k^2.$$

Furthermore, let $D = BB^*$ and $D1 = D\{1, 2, \cdots, i-$

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$1, i+1, \cdots, k\}$ which has the same meaning as $B1$, then $D1 = B1B1^* + zz^T$, where

$$z = (b_{i1}, b_{i,i-1}, b_{i,i+1}, \cdots, b_{ik})^T.$$

And $K_1^2, K_2^2, \cdots, K_{k-1}^2$ are the eigenvalues or $D1$ in increasing order. $K_i \geqslant 0, i = 1, 2, \cdots, k-1$. According to Cauchy interlacing theorem[1], we get

$$W_1^2 \leqslant K_1^2 \leqslant \cdots \leqslant K_{k-1}^2 \leqslant W_k^2$$

or

$$W_1 \leqslant K_1 \leqslant \cdots \leqslant K_{k-1} \leqslant W_k.$$

On the other hand, let the eigenvalues of $B1B1^*$ be $\theta_1^2, \theta_2^2, \cdots, \theta_{k-1}^2$ in increasing order, where $\theta_i \geqslant 0, i = 1, 2, \cdots, k-1$. According to Theorem 2, we get

$$\theta_1^2 \leqslant K_1^2 \leqslant \theta_2^2 \leqslant \cdots \leqslant \theta_{k-1}^2 \leqslant K_{k-1}^2$$

or

$$\theta_1 \leqslant K_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_{k-1} \leqslant K_{k-1}.$$

Remark 1 For convenience we call the conclusion of Theorem to be generalized interlacing theorem. Furthermore, these inequalities can not be combined into one. For example,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, A1 = [0],$$

where $W_1 = W_2 = 1, \theta_1 = 0$.

That is, it can not be showed that they are in an inequality as Cauchy interlacing theorem, but they can be described as generalized interlacing theorem.

Now let us use the result in Theorem 3 to get the following result.

Theorem 3 Let $B \in R^{k \times k}$ and B be normal and the submatrix $B1$, which is obtained by deleting some row and the same column of B, is normal. $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of B in modulus increasing order and $\{\_{i}\}_{i=1}^{n-1}$ are the eigenvalues of $B1$ in modulus increasing order. Then there exists the sequence of nonnegative real numbers $\{K_i\}_{i=1}^{k-1}$ which satisfies

$$|\lambda_1| \leqslant K_1 \leqslant |\lambda_2| \leqslant \cdots \leqslant |\lambda_{n-1}| \leqslant K_{n-1} \leqslant |\lambda_n|$$

and

$$|\_{1}| \leqslant K_1 \leqslant |\_{2}| \leqslant \cdots \leqslant |\_{n-1}| \leqslant K_{n-1}.$$

Proof Since that the modulus of the eigenvalues are just the singular values, we can get the conclusion according to Theorem 3.

Apparently, the theorem fits to Hermitian and skew-Hermitian matrixes. And the normal matrix plays an important role not only in economics but also in physics.

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实验表明，WBLE具有很多优点，采用 WBLE结构后，输入自相关矩阵很接近一个对角阵，通过能量规一化，可以使 WBLE的收敛速度比传统的 LE快很多，而误码性能则与 LE相同。当信道特性发生变化时，WBLE能够迅速跟踪这一变化。在很多情况下 WBLE的权系数中只有少数具有较大的能量，而其它系数能量则小得多。

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