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On Some Results of Harmonic Morphisms between Semi-Euclidean Spaces*

关于半欧氏空间之间调和同态的一些结果

Lu Weijun
卢 卫君

(Dept. of Math., Guangxi University for Nationalities, Xixiangtang, Nanning, Guangxi, 530006, China)
(广西民族学院数学系 南宁市西乡塘 530006)

Abstract Harmonic morphisms between semi-Euclidean spaces $R^m_r \to R^n_s$ is studied. Two theorems are given to construct harmonic morphisms between semi-Euclidean spaces in two ways, and Ou Yielin's corresponding results are generalized. Some interesting examples of quadratic harmonic morphisms between semi-Euclidean spaces are presented.

Key words semi-Riemanian manifolds, harmonic maps, harmonic morphisms

摘要 研究半欧氏空间 $R^m_r \to R^n_s$ 之间的调和同态. 推广了欧业林构造欧氏空间之间调和同态的方法，得出半欧氏空间之间2个相应的定理. 同时，给出半欧氏空间之间二次调和同态的一些有趣的例子.

关键词 半黎曼流形 调和映射 调和同态

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The notions of harmonic morphisms and horizontally weakly conformal maps between Riemannian manifolds were studied in the context of differential geometry in the late 70s of last century. A harmonic morphism is characterized as a horizontally weakly conformal and harmonic map independently in References [1, 2]. For a detailed account on harmonic maps and harmonic morphisms between Riemannian manifolds, we refer to references including References [3~6]. Recently, the corresponding notions for semi-Riemannian manifolds have been studied by Parmar[7]. We refer to O'Neill[8] concerning semi-Riemannian manifolds (where the metric tensor may be indefinite and hence the Laplace-Betrami operator may not be elliptic).

This paper comprises two sections. In Section 1 we introduce and recall some fundamental concepts and facts concerning semi-Riemannian manifolds and harmonic maps, harmonic morphisms and horizontally weakly conformal maps between these spaces. In Section 2 we prove our main theorems and give an example of nontrivial harmonic morphism $\varphi: R^4_2 \to R^3_2$ and show that its complete lift is a quadratic harmonic morphism $H: R^4_2 \times R^4_2 \to R^3_2$.

The interesting problem of constructing and classifying polynomial harmonic morphisms between Euclidean spaces have been studied extensively and the background information could be obtained from References [3, 9~16].

1 Preliminaries

1.1 Semi-Riemannian manifolds

Definition 1 A semi-Riemannian manifold M is a C^∞-manifold endowed with a metric g_M, i.e. a symmetric non-degenerate (0, 2) tensor field on M, with constant indices of positivity and negativity $\mathrm{ind}_+ M$ and $\mathrm{ind}_- M$, respectively. The non-degeneracy means that $\mathrm{ind}_+ M + \mathrm{ind}_- M = \dim M$.

Definition 2 A subspace of U of the tangent space $T_x M, x \in M$, is called non-degenerate if the restriction of g_x to $U \times U$ is non-degenerate, that is, if 0 is the only vector $X \in U$ such that $g_x(X, Y) = 0$ for

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every $Y \in U$; Otherwise U is called degenerate.

Let $\varphi: M \to N$ be a C^1-map between semi-Riemannian manifolds M and N of dimensions m and n, respectively. For each $x \in M$ we consider the following two subspaces $K_x(\varphi) = K_x$ and $K_x^{\perp}(\varphi) = K_x^{\perp}$ of $T_x M$.

$K_x = \mathrm{Ker} d\varphi(x) = \{x \in T_x M \mid d\varphi(x)(X) = 0\}$,
$K_x^{\perp} = \{X \in T_x M \mid g_x(X, Y) = 0, \text{for very } Y \in K_x\}$.

In the Riemannian case, we have $K_x \oplus K_x^{\perp} = T_x M$, and it is customary to call K_x the vertical space and $K_x^{\perp}$ the horizontal space at x. However, in the semi-Riemannian case, we can not call $K_x^{\perp}$ the orthogonal complement of K_x since $K_x + K_x^{\perp}$ is generally not all of $T_x M$: $K_x + K_x^{\perp} \neq T_x M$, or equivalently: $K_x \cap K_x^{\perp} \neq \{0\}$; this is further equivalent to K_x being degenerate.

1.2 Horizontally weakly conformal maps

Definition 3 Let $\varphi: M \to N$ be a non-degenerate map of semi-Riemannian such that at points $x \in M$ where $d\varphi_x \neq 0$, $d\varphi|_{K_x^{\perp}}: K_x^{\perp} \to T_{\varphi(x)} N$ is conformal and surjective, i.e. there is a continuous function $\lambda: M \to R$ such that

$$\langle d\varphi(X), d\varphi(Y) \rangle = \lambda^2 \langle X, Y \rangle_M, \forall X, Y \in K_x^{\perp}.$$

At the critical points of φ, i.e. points $x \in M$ where $d\varphi = 0$, we put $\lambda = 0$; then λ^2 is smooth. We call such a map ϕ horizontally weakly conformal with dilation λ.

Remark 1 (1) A map $\varphi: M \to N$ is non-degenerate, if its fibres $\varphi^{-1}(q), q \in N$ are semi-Riemannian submanifolds of M, or $K_x(\varphi)$ is non-degenerate for every $x \in M$. (2) The term "weakly" refers to the possible occurrence of point $x \in M$ at which $\lambda(x) = 0$.

Lemma 1[17] A C^1-map $\varphi: M \to N$ is horizontally weakly conformal with dilation λ if and only if

$$\langle \nabla_M \varphi^{\alpha}, \nabla_M \varphi^{\beta} \rangle_M = \lambda^2 (g_N^{\alpha\beta} \circ \varphi),$$

where $(y^1, y^2, \cdots, y^n)$ are local coordinates in N, and ∇_M denote the gradient operator for the manifold M, and $\alpha, \beta = 1, 2, \cdots, n$.

1.3 Semi-Euclidean space

Definition 4 The index ν of a symmetric bilinear g on M is the dimension of the largest subspace $U \subset M$ on which $g|_U$ is negative definite.

The constant ν of index g_x on a semi-Riemannian manifold M is called the index of M: $0 \leqslant \nu \leqslant m = \dim M$. If $\nu = 0$, M is a Riemannian manifold, each g_x is then a positive definite inner product on $T_x M$.

If $(x^1, \cdots, x^m)$ is a coordinate system on $U \subset M$, then the components of the metric tensor g on U are

$$g_{ij} = < \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} >, \text{ for } 1 \leqslant i, j \leqslant m.$$

Thus for vector fields $X = \sum X^i \frac{\partial}{\partial x^i}$ and $Y = \sum Y^i \frac{\partial}{\partial x^i}$

$$g(X, Y) = < X, Y > = \sum g_{ij} X^i Y^j.$$

Since g is non-degenerate, at each point x of U the matrix $g_{ij}(x)$ is invertible, and its inverse matrix is denoted by $g_{ij}(x)$. Finally on U the metric tensor can be written as $g = \sum g_{ij} dx^i \otimes dx^j$.

Recall that for each $p \in R^m$ there is a canonical linear isomorphism between R^m and $T_p(R^m)$ through which we can identify $T_p(R^n)$ with R^m. Under this identification the inner product on R^m gives rise to a metric tensor on R^m by

$$\langle X_p, Y_p \rangle = X_p \cdot Y_p = \sum_{i=1}^{m} X^i Y^i.$$

Henceforth in any geometric context R^m will denote the resulting Riemannian manifold, called Euclidean m-space. For an integer ν with $0 \leqslant \nu \leqslant m$, changing the first ν plus signs above to minus gives a metric tensor

$$\langle X_p, Y_p \rangle = -\sum_{i=1}^{\nu} X^i Y^i + \sum_{i=\nu+1}^{m} X^i Y^i$$

of index ν. The resulting manifold is called semi-Euclidean m-space, which is denoted R^m_ν. If $\nu = 0$, R^m_ν then reduces to R^m. Using the notation

$$\varepsilon_i = \begin{cases} -1, & \text{for } 1 \leqslant i \leqslant \nu \\ +1, & \text{for } \nu+1 \leqslant i \leqslant m \end{cases}$$

then the metric tensor of R^m_ν can be written as

$$g = \sum \varepsilon_i dx^i \otimes dx^i$$

The norm $|X_p|$ of a tangent vector $X_p \in T_p$ is $|X_p| = | < X_p, X_p > |^{1/2}$, and unit vectors, orthogonality, and orthonormality are as the case of Riemannian manifold.

1.4 Harmonic maps and morphisms

For a semi-Riemannian manifold (M, g) the Laplace-Beltrami operator $\triangle_M$ is given, in local coordinates $(x^i)_{x=1}^m$, by

$$\triangle_M = \frac{1}{\sqrt{|\det(g_{ij})|}} \sum_i \frac{\partial}{\partial x^i} \left(\sqrt{|\det(g_{ij})|} \sum_j g^{ij} \frac{\partial}{\partial x^j} \right).$$

Although the operator $\triangle_M$ is not elliptic when ind $M > 0$ and ind $-M > 0$, we still keep the definition of harmonic function $f: M \to R$ as a C^2 local solutions to the Laplace-Beltrami equation $\triangle_M f = 0$.

Definition 5 A tension field $T(\varphi)$ of a C^2-map $\varphi: M \to N$ between semi-Riemannian manifolds is a vector field along φ which to each point $x \in M$ assigns the tangent vector, denoted $T(\varphi)(x) \in T_{\varphi(x)} N$, whose contravariant components $T^k(\varphi)(x)$ in terms of local coordinates $(y^1, \cdots, y^n)$ in N, can written as

$$T^k(\varphi) = \triangle_M \varphi^k + \sum_{\alpha,\beta=1}^{n} g_x(\nabla_M \varphi^{\alpha}, \nabla_M \varphi^{\beta})(\Gamma^k_{\alpha\beta} \circ \varphi),$$

here $\Gamma^{\gamma}_{\alpha\beta}$ denote the Christoffel symbols for the target manifold N. φ is called a harmonic map if $T(\varphi)=0$.

Definition 6 A C^2-map between semi-Riemannian manifolds is called a harmonic morphism, if whenever f is a harmonic function on an open set $V\subset N$ and $\varphi^{-1}(V)$ is non-empty, then the pull-back of φ is harmonic on $\varphi^{-1}(V)\subset M$.

Theorem 1 A map $\varphi:(M,g)\to R^n_s$ is a non-generate harmonic morphism if and only if φ is both a harmonic map and a horizontally weakly conformal map.

For a proof see Reference [7].

Definition 7 A polynomial map $\varphi: R^m_r\to R^n_s$ is a map with polynomial components $\varphi^1,\cdots,\varphi^n$. Its degree is the maximal degree of the component polynomials.

A polynomial harmonic morphism φ means that φ is both a harmonic morphism and a polynomial map.

2 Proofs of the Main Results

First of all we prove the following.

Lemma 2 For a map $\varphi: R^m_r\supset U\to R^n_s$ between semi-Euclidean space with $\varphi(x)=(\varphi^1(x),\cdots,\varphi^n(x))$, the harmonicity and horizontally weakly conformlity are just equivalent to the following conditions respectively:

$$-\sum_{i=1}^{r}\frac{\partial^2\varphi^\alpha}{\partial x^{i2}}+\sum_{i=r+1}^{m}\frac{\partial^2\varphi^\alpha}{\partial x^{i2}}=0, \qquad (1)$$

$$-\sum_{i=1}^{r}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^i}+\sum_{i=r+1}^{m}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^i}=\lambda^2\varepsilon_\alpha\delta_{\alpha\beta}, \qquad (2)$$

where

$$\alpha,\beta=1,2,\cdots,n;$$

$(x^1,\cdots,x^m)$ are standard coordinates of R^m_r; $\lambda: R^m_r\to R$ is the dilation of φ;

$$\varepsilon_\alpha=\begin{cases}-1, & \alpha=1,2,\cdots,s;\\ 1, & \alpha=s+1,\cdots,n.\end{cases}$$

Proof By Definition 5, we see that

$$T(\varphi)=0,$$

and for every $\alpha=1,2,\cdots,n$,

$$T^\alpha(\varphi)=\triangle_M\varphi^\alpha+\sum_{\beta,\gamma}^{m}g^M(\nabla_M\varphi^\beta,\nabla_M\varphi^\gamma)(\Gamma^\alpha_{\beta\gamma}\circ\varphi)=0,$$

where $\Gamma^\alpha_{\beta\gamma}$ vanish in R^n_s. Thus

$$\triangle_M\varphi^\alpha=0.$$

This implies that every φ^α is a harmonic function. Finally by the definition of $\triangle_M$ in Section 1, we get Equality (1):

$$-\sum_{i=1}^{r}\frac{\partial^2\varphi^\alpha}{\partial x^{i2}}+\sum_{i=r+1}^{m}\frac{\partial^2\varphi^\alpha}{\partial x^{i2}}=0.$$

On the other hand, we start from

$$<\nabla_M\varphi^\alpha,\nabla_M\varphi^\beta>_M=\lambda^2(g_N^{\alpha\beta}\circ\varphi),$$

then by Lemma 1 we get

$$\sum_{i,j}^{m}\varepsilon_i\delta_{ij}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}=\lambda^2\varepsilon_\alpha\delta_{\alpha\beta}.$$

i.e. $$-\sum_{i=1}^{r}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^i}+\sum_{i=r+1}^{m}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^i}=\lambda^2\varepsilon_\alpha\delta_{\alpha\beta}.$$

Henceforth Lemma 2 follows.

Lemma 3 The complete lift of a harmonic map $\varphi: R^m_r\to R^n_s$ is again a harmonic map.

Proof Let $\varphi: R^m_r\to R^n_s$, $\varphi(x)=(\varphi^1(x),\cdots,\varphi^n(x))$, be a harmonic map, then by Lemma 2 we have

$$-\sum_{i=1}^{r}\frac{\partial^2\varphi^\alpha}{\partial x^{i2}}+\sum_{i=r+1}^{m}\frac{\partial^2\varphi^\alpha}{\partial x^{i2}}=0,$$

where

$$\alpha=1,2,\cdots,n.$$

We next check that

$$\Phi(x,y)=\Big(\sum_{i=1}^{m}\frac{\partial\varphi^1}{\partial x^i}y^i,\cdots,\sum_{i=1}^{m}\frac{\partial\varphi^n}{\partial x^i}y^i\Big),$$

$$\Phi^\alpha(x,y)=\sum_{i=1}^{m}\frac{\partial\varphi^\alpha}{\partial x^i}y^i,$$

is a harmonic map.

Let

$$u^1=x^1,\cdots,u^m=x^m,u^{m+1}=y^1,\cdots,u^{2m}=y^m;$$

$$\varepsilon_k=\begin{cases}-1, & k=1,2,\cdots,r,m+1,\cdots,m+r;\\ 1, & k=r+1,\cdots,m,m+r+1,\cdots,2m.\end{cases}$$

But

$$\sum_{k=1}^{2m}\varepsilon_k\frac{\partial^2\Phi^\alpha}{\partial(u^k)^2}=\sum_{k=1}^{2m}\varepsilon_k\frac{\partial^2}{\partial(u^k)^2}\Big(\sum_{i=1}^{m}\frac{\partial\varphi^\alpha}{\partial x^i}y^i\Big)$$
$$=\sum_{j=1}^{m}\varepsilon_j\frac{\partial^2}{\partial(x^j)^2}\Big(\sum_{i=1}^{m}\frac{\partial\varphi^\alpha}{\partial x^i}y^i\Big)+\sum_{j=1}^{m}\varepsilon_j\cdot\frac{\partial^2}{\partial(y^j)^2}\Big(\sum_{i=1}^{m}\frac{\partial^2\varphi^\alpha}{\partial x^i}y^i\Big)=\sum_{i=1}^{m}y^i\frac{\partial}{\partial x^i}\cdot$$
$$\Big(-\sum_{j=1}^{r}\frac{\partial^2\varphi^\alpha}{\partial(x^j)^2}+\sum_{j=r+1}^{m}\frac{\partial^2\varphi^\alpha}{\partial(x^j)^2}\Big)+0$$
$$=0.$$

This ends the proof.

Theorem 2 Let $\varphi: R^m_r\to R^n_s$ be a harmonic morphism defined by homogeneous polynomial of degree 2, then the complete lift of φ, defined by $\Phi: R^m_r\times R^m_r\to R^n_s$ with $\Phi(x,y)=\Big(\sum_{i=1}^{m}\frac{\partial\varphi^1}{\partial x^i}y^i,\cdots,\sum_{i=1}^{m}\frac{\partial\varphi^n}{\partial x^i}y^i\Big)$ is a harmonic morphism, where $R^m_r\times R^m_r$ is given the product metric having the form $\sum_{k=1}^{2m}\varepsilon_k dx^k\otimes dx^k$.

Proof Now the harmonicity of $\Phi(x,y)$ follows from that of φ by Lemma 3. So we only need to check that $\Phi(x,y)$ is a horizontally weakly conformal map. Now that φ is a homogeneous polynomial of degree 2, we can write

$$\varphi(x) = (x A_1 x^t, \cdots, x A_n x^t),$$

where $x = (x^1, \cdots, x^m)$, x^t denotes the transpose of x and A_α ($\alpha = 1, 2, \cdots, n$) is a symmetric matrix of $m \times m$. Thus, we can write $A_\alpha = (A_\alpha^1, \cdots, A_\alpha^m) = ((A_\alpha^1)^t, \cdots, (A_\alpha^m)^t)^t$, where A_α^i denotes the i th column vector of A_α, and $(A_\alpha^i)^t$ the i th row vector of A_α.

A routine calculation gives

$$\frac{\partial \varphi^\alpha}{\partial x^i} = 2x A_\alpha^i = 2(A_\alpha^i)^t x^t, \qquad (3)$$

$$\psi^\alpha(x,y) = \sum_{i=1}^m \frac{\partial \varphi^\alpha}{\partial x^i} y^i = \left(\frac{\partial \varphi^\alpha}{\partial x^1}, \cdots, \frac{\partial \varphi^\alpha}{\partial x^m}\right) y^t = 2x A_\alpha y^t,$$

$$\frac{\partial \psi^\alpha(x,y)}{\partial x^i} = 2(A_\alpha^i)^t y^t = 2y A_\alpha^i, \qquad (4)$$

$$\frac{\partial \psi^\alpha(x,y)}{\partial y^i} = 2x A_\alpha^i = 2(A_\alpha^i)^t x^t. \qquad (5)$$

On the other hand, since φ is a horizontally weakly conformal map by Theorem 1, we have Equality (2) in Lemma 2. Together with Equality (3), we can get

$$-4\sum_{i=1}^r (x A_\alpha^i)(x A_\beta^i) + 4\sum_{i=r+1}^m (x A_\alpha^i)(x A_\beta^i) = \lambda_1^2 \varepsilon_\alpha \delta_{\alpha\beta}, \qquad (6)$$

$$-4\sum_{j=1}^r (y A_\alpha^j)(y A_\beta^j) + 4\sum_{j=r+1}^m (y A_\alpha^j)(y A_\beta^j) = \lambda_2^2 \varepsilon_\alpha \delta_{\alpha\beta}. \qquad (7)$$

Thus, we have

$$\sum_{k=1}^{2m} \varepsilon_k \frac{\partial \psi^\alpha}{\partial u^k} \frac{\partial \psi^\beta}{\partial u^k} = -\sum_{i=1}^r \frac{\partial \psi^\alpha}{\partial x^i}\frac{\partial \psi^\beta}{\partial x^i} + \sum_{i=r+1}^m \frac{\partial \psi^\alpha}{\partial x^i}\frac{\partial \psi^\beta}{\partial x^i} - \sum_{j=1}^r \frac{\partial \psi^\alpha}{\partial y^j}\frac{\partial \psi^\beta}{\partial y^i} + \sum_{j=r+1}^m \frac{\partial \psi^\alpha}{\partial y^j}\frac{\partial \psi^\beta}{\partial y^j} = -4\sum_{i=1}^r (x A_\alpha^i)(x A_\beta^i) +$$

$$4\sum_{i=r+1}^m (x A_\alpha^i)(x A_\beta^i) - 4\sum_{j=1}^r (y A_\alpha^j)(y A_\beta^j) +$$

$$4\sum_{j=r+1}^m (y A_\alpha^j)(y A_\beta^j) \qquad \text{(by (4) and (5))}$$

$$= (\lambda_1^2 + \lambda_2^2)\varepsilon_\alpha \delta_{\alpha\beta}. \qquad \text{(by (6) and (7))}$$

So it is easily to see that $\psi(x,y)$ is a horizontally weakly conformal map. Thus we have proved Theorem 2.

Theorem 3 Let $\varphi_1: M_r^m \to R_s^n$ and $\varphi_2: N_q^p \to R_s^n$ be two harmonic morphisms, then the direct sum $\varphi_1 \oplus \varphi_2$ of φ_1 and φ_2, defined by $\varphi_1 \oplus \varphi_2: M_r^m \times N_q^p \to R_s^n$ with $(\varphi_1 \oplus \varphi_2)(x,y) = \varphi_1(x) + \varphi_2(y)$ is a harmonic morphism, where the product manifold $M_r^m \times N_q^p$ is provided with the product metric.

Proof Let

$$(x^1, x^2, \cdots, x^m, y^1, y^2, \cdots, y^p),$$
$$(x^1, x_2, \cdots, x^m),$$

and

$$(y^1, y^2, \cdots, y^p)$$

be a coordinate system on $U \times V \subset M_r^m \times N_q^p$, $U \subset M_r^m$ and $V \subset N_q^p$, respectively.

At first we have by Lemma 2

$$-\sum_{i=1}^r \frac{\partial^2 \varphi_1^\alpha(x)}{\partial x^{i^2}} + \sum_{i=r+1}^m \frac{\partial^2 \varphi_1^\alpha(x)}{\partial x^{i^2}} = 0. \qquad (8)$$

$$-\sum_{i=1}^r \frac{\partial \varphi_1^\alpha(x)}{\partial x^i}\frac{\partial \varphi_1^\beta(x)}{\partial x^i} + \sum_{i=r+1}^m \frac{\partial \varphi_1^\alpha(x)}{\partial x^i}\frac{\partial \varphi_1^\beta(x)}{\partial x^i} = \lambda_1^2 \varepsilon_\alpha \delta_{\alpha\beta}. \qquad (9)$$

$$-\sum_{j=1}^q \frac{\partial^2 \varphi_2^\alpha(y)}{\partial y^{j^2}} + \sum_{j=q+1}^p \frac{\partial^2 \varphi_2^\alpha(y)}{\partial y^{j^2}} = 0. \qquad (10)$$

$$-\sum_{j=1}^q \frac{\partial \varphi_2^\alpha(y)}{\partial y^j}\frac{\partial \varphi_2^\beta(y)}{\partial y^j} + \sum_{j=q+1}^p \frac{\partial \varphi_2^\alpha(y)}{\partial y^j}\frac{\partial \varphi_2^\beta(y)}{\partial y^j} = \lambda_2^2 \varepsilon_\alpha \delta_{\alpha\beta}. \qquad (11)$$

So we get

$$\sum_{k=1}^{m+p} \varepsilon_k \frac{\partial^2 (\varphi_1 \oplus \varphi_2)^\alpha(x,y)}{\partial (u^k)^2}$$
$$= \sum_{k=1}^m \varepsilon_k \frac{\partial^2 (\varphi_1^\alpha(x) + \varphi_2^\alpha(y))}{\partial (x^k)^2} + \sum_{j=1}^p \varepsilon_j \frac{\partial^2 (\varphi_1^\alpha(x) + \varphi_2^\alpha(y))}{\partial (y^j)^2}$$
$$= -\sum_{i=1}^r \frac{\partial^2 \varphi_1^\alpha(x)}{\partial x^{i^2}} + \sum_{i=r+1}^m \frac{\partial^2 \varphi_1^\alpha(x)}{\partial x^{i^2}}$$
$$-\sum_{j=1}^q \frac{\partial^2 \varphi_2^\alpha(y)}{\partial y^{j^2}} + \sum_{j=q+1}^p \frac{\partial^2 \varphi_2^\alpha(y)}{\partial y^{j^2}}$$
$$= 0, \text{ (by (8) and (10))}$$

here we denote that

$$u^1 = x^1, \cdots, u^m = x^m, u^{m+1} = y^1, \cdots, u^{m+p} = y^p;$$

and

$$\varepsilon_k = \begin{cases} -1, & k = 1, 2, \cdots, r, m+1, \cdots, m+q; \\ 1, & k = r+1, \cdots, m, m+q+1, \cdots, m+p. \end{cases}$$

Again,

$$\sum_{k=1}^{m+p} \varepsilon_k \frac{\partial(\varphi_1(x) \oplus \varphi_2(y))^\alpha}{\partial u^k}\frac{\partial(\varphi_1(x) \oplus \varphi_2(y))^\beta}{\partial u^k}$$
$$= \sum_{k=1}^m \varepsilon_k \frac{\partial(\varphi_1^\alpha(x) + \varphi_2^\alpha(y))}{\partial(x^k)}\frac{\partial(\varphi_1^\beta(x) + \varphi_2^\beta(y))}{\partial(x^k)}$$
$$+ \sum_{j=1}^p \varepsilon_j \frac{\partial(\varphi_1^\alpha(x) + \varphi_2^\alpha(y))}{\partial(y^j)}\frac{\partial(\varphi_1^\beta(x) + \varphi_2^\beta(y))^\alpha}{\partial(y^j)}$$
$$= -\sum_{i=1}^r \frac{\partial \varphi_1^\alpha(x)}{\partial x^i}\frac{\partial \varphi_1^\beta(x)}{\partial x^i} + \sum_{i=r+1}^m \frac{\partial \varphi_1^\alpha(x)}{\partial x^i}\frac{\partial \varphi_1^\beta(x)}{\partial x^i}$$
$$-\sum_{j=1}^q \frac{\partial \varphi_2^\alpha(y)}{\partial y^j}\frac{\partial \varphi_2^\beta(y)}{\partial y^j} + \sum_{j=q+1}^p \frac{\partial \varphi_2^\alpha(y)}{\partial y^j}\frac{\partial \varphi_2^\beta(y)}{\partial y^j}$$
$$= \lambda_1^2 \varepsilon_\alpha \delta_{\alpha\beta} + \lambda_2^2 \varepsilon_\alpha \delta_{\alpha\beta} \text{ (by (9) and (11))}$$
$$= (\lambda_1^2 + \lambda_2^2)\varepsilon_\alpha \delta_{\alpha\beta}.$$

This ends the proof of Theorem 3.

Example 1 Let

$$\varphi: R_2^4 \to R_2^3$$

by

$$\varphi(x^1, x^2, x^3, x^4) = (2x^1x^3 - 2x^2x^4, 2x^1x^4 + 2x^2x^3, (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2).$$

Then φ is a harmonic morphism defined by homogeneous polynomial of degree 2 with dilation $4|x|^2$.

In fact, since

$\varphi_1 = 2x^1x^3 - 2x^2x^4, \varphi_2 = 2x^1x^4 + 2x^2x^3, \varphi_3 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2,$

now by simply calculating, we get

$$\sum_{i=1}^{2} \frac{\partial^2 \varphi_\alpha}{\partial x^{i2}} + \sum_{i=3}^{4} \frac{\partial^2 \varphi_\alpha}{\partial x^{i2}} = 0, \alpha = 1, 2, 3.$$

$$-\sum_{i=1}^{2} \frac{\partial \varphi_\alpha}{\partial x^i}\frac{\partial \varphi_\beta}{\partial x^i} + \sum_{i=3}^{4} \frac{\partial \varphi_\alpha}{\partial x^i}\frac{\partial \varphi_\beta}{\partial x^i} = 0, \alpha \neq \beta$$

$$-\sum_{i=1}^{2} \left(\frac{\partial \varphi_\alpha}{\partial x^i}\right)^2 + \sum_{i=3}^{4} \left(\frac{\partial \varphi_\alpha}{\partial x^i}\right)^2$$

$$= \begin{cases} -4|x|^2 = 4|x|^2 \varepsilon_1 \eta_{11}, \alpha = 1, \\ -4|x|^2 = 4|x|^2 \varepsilon_2 \eta_{22}, \alpha = 2, \\ 4|x|^2 = 4|x|^2 \varepsilon_3 \eta_{33}, \alpha = 3. \end{cases}$$

By Lemma 2, φ is exactly a harmnic morphism.

Example 2 As in Example 1, map $\varphi: R^m_r \to R^n_s$ is a harmonic morphism, then the complete lift of φ defined by $\Phi: R^4_2 \times R^4_2 \to R^3_2$ is a harmonic morphism with dilation $4(|x|^2 + |y|^2)$.

In fact,

$$\Phi(x, y) = \left(\sum_{i=1}^{4} \frac{\partial \varphi_1(x)}{\partial x^i} y^i, \sum_{i=1}^{4} \frac{\partial \varphi_2(x)}{\partial x^i} y^i, \sum_{i=1}^{4} \frac{\partial \varphi_3(x)}{\partial x^i} y^i\right),$$

where

$\varphi_1 = 2x^1x^3 - 2x^2x^4, \varphi_2 = 2x^1x^4 + 2x^2x^3, \varphi_3 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2.$

So

$$\Phi^1(x, y) = 2x^3y^1 - 2x^4y^2 + 2x^1y^3 - 2x^2y^4,$$
$$\Phi^2(x, y) = 2x^4y^1 + 2x^3y^2 + 2x^2y^3 + 2x^1y^4,$$
$$\Phi^3(x, y) = 2x^1y^1 + 2x^2y^2 + 2x^3y^3 + 2x^4y^4.$$

It can be calculated that

$$\sum_{k=1}^{8} \varepsilon_k \frac{\partial^2 \Phi^\alpha}{\partial (u^k)^2} = \sum_{k=1}^{4} \varepsilon_k \frac{\partial^2 \Phi^\alpha}{\partial (x^k)^2} + \sum_{j=1}^{4} \varepsilon_j \frac{\partial^2 \Phi^\alpha}{\partial (y^j)^2} = 0, \quad \alpha = 1, 2, 3.$$

$$\sum_{k=1}^{8} \varepsilon_k \frac{\partial \Phi^\alpha}{\partial u^k}\frac{\partial \Phi^\beta}{\partial u^k} = \sum_{j=1}^{4} \varepsilon_j \frac{\partial \Phi^\alpha}{\partial x^j}\frac{\partial \Phi^\beta}{\partial x^j} + \sum_{j=1}^{4} \varepsilon_j \frac{\partial \Phi^\alpha}{\partial y^j}\frac{\partial \Phi^\beta}{\partial y^j} = 0, \quad \alpha \neq \beta$$

$$\sum_{k=1}^{8} \varepsilon_k \left(\frac{\partial \Phi^\alpha}{\partial u^k}\right)^2 = \sum_{j=1}^{4} \varepsilon_j \left(\frac{\partial \Phi^\alpha}{\partial x^j}\right)^2 + \sum_{j=1}^{4} \varepsilon_j \left(\frac{\partial \Phi^\alpha}{\partial y^j}\right)^2$$

$$= \begin{cases} -4(|x|^2 + |y|^2) = 4(|x|^2 + |y|^2) \varepsilon_1 \eta_{11}, \alpha = 1, \\ -4(|x|^2 + |y|^2) = 4(|x|^2 + |y|^2) \varepsilon_2 \eta_{22}, \alpha = 2, \\ 4(|x|^2 + |y|^2) = 4(|x|^2 + |y|^2) \varepsilon_3 \eta_{33}, \alpha = 3. \end{cases}$$

This amounts to check that $\Phi(x, y)$ is a harmonic morphism.

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