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A Formula for Computing the Homeomorphism Class Number of G. M. $_{n+1\ \text{vertices}}$

计算图式流形 $_{n+1\ \text{vertices}}$ 同胚类个数的一个公式

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Abstract The formula for computing the homeomorphism classes number of graphlike manifold $_{n+1\ \text{vertices}}$ is developed by finding number of sequences.

Key words negative edge, distribution of negative edges of the n-polygon, sequence of number of nonbeginning, nonending and infinite cyclic, graphlike manifold

摘要 利用求数列个数办法推导出计算图式流形 $_{n+1\ \text{vertices}}$ 的同胚类个数公式.

关键词 负边 n边形的负边分布 无首无尾无限 循环数列 图式流形
中图法分类号 O157.5; O189

It has been known that the homeomorphism classes of G. M. , G. M. and G. M. , and the homeomorphic classification of G. M. $_{n+1\ \text{vertices}}$ be equal to the homeomorphic classification of $[\frac{n}{2}]$ negative edges ($n \geqslant 6$). How many homeomorphism classes of G. M. $_{n+1\ \text{vertices}}$ are there? It is to be answered in this paper.

In G. M. $_{n+1\ \text{vertices}}$, if a radioedge is negative and twisted out of its vertex, it is set positive. So the homeomorphism class of G. M. $_{n+1\ \text{vertices}}$ is equal to the "distribution of negative edges of the n-polygon" (since a homeomorphism must be a local homeomorphism, distinct negative-edge-distribution corresponds to distinct homeomorphism class of G. M. $_{n+1\ \text{vertices}}$ $n > 3$.).

Suppose each edge is either positive or negative in the polygon, the case of negative-edge-distribution is that of position of positive edges relative to negative edges. When there are k negative edges, there are also $n-k$ positive edges. It is taken into consideration that how many positive edges there are among negative edges.

Suppose

$$n-k = p = p_1 + p_2 + \cdots + p_k, P_i \leqslant p_j, i < j. \quad (1)$$

p_i is nonnegative integer, $i = 1, 2, \cdots, k$, showing the number of positive edges which are between two "nearest" negative edges, the formula (1) is regarded as a divided formula of p, in which a random permutation of p_i or an ordinal array of p_i is corresponding to a distribution of negative edges, e. g. when $n = 10, k = 4, n-k = 6$, $6 = 0 + 0 + 2 + 4$ is a divided formula corresponding to the negative-edge-distribution as showed in Figure 1 (the bar on edge denotes negative edge). It should be paid attention to that the divided formula $6 = (0, 0, 2, 4)$ is corresponding to the same distribution of negative edges as arrays (0, 2, 4, 0), (2, 4, 0, 0), (4, 0, 0, 2), (0, 0, 4, 2), (0, 4, 2, 0), (4, 2, 0, 0), (2, 0, 0, 4), but not to the same one as (0, 2, 0, 4). Equality $6 = (1, 1, 1, 3)$ is another divided formula (when $n = 10, k = 4$), but Equality $6 = (1, 1, 3, 1)$ is not a divided formula, only a permutation of p_i in (1, 1, 1, 3).

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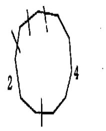


图 1
Fig 1

Let us consider (A) how many divided formulae of p there are, and (B) how many real different permutations from a divided formula of p there are. The former can be easily answered, e. g. $n - k = 10 - 4 = 6 = (0, 0, 0, 6) = (0, 0, 1, 5) = (0, 0, 2, 4) = (0, 0, 3, 3) = (0, 1, 1, 4) = (0, 1, 2, 3) = (1, 1, 1, 3) = (1, 1, 2, 2)$. But the later is not.

In a regular n-polygon, the distribution number of k negative edges is corresponding to the number of cyclic sequences below which are non-beginning, non-ending and infinite:

$$\dot{q}_1 q_2 \cdots q_{k-1} \dot{q}_k = \cdots q_1 q_2 \cdots q_k q_1 q_2 \cdots q_k \cdots , \qquad (2)$$

in which

① $\sum_{i=1}^{k} q_i = p, q_i$ –nonnegative integer,

② appoint $\dot{q}_1 \dot{q}_2 \cdots \dot{q}_k = \dot{q}_k q_{k-1} \cdots \dot{q}_1$.

When $\exists p_i$ is different from $p_j, \forall j \neq i$, suppose the divided formula (1) of p was p_k, and fixed. If $\forall r \neq s, r, s \in \{1, 2, \cdots, k-1\}, p_r \neq p_s$, the number of permutation of $p_j, j = 1, 2, \cdots, k-1$ on the $k-1$ positions is $(k-1)!$. According to ② , every permutation, except for $p_1, p_2, \cdots, p_{k-1}$ which is symmetry (i. e. $p_i = p_{k-i}, i = 1, 2, \cdots, k-1$), counts twice, e. g. $(p_1, p_2, \cdots, p_{k-1}, p_k)$ and $(p_{k-1}, \cdots, p_2, p_1, p_k)$. When $\exists p_j = p_j, i \neq j, i \neq k, j \neq k$, i. e. $(p_1, p_2, \cdots, p_{k-1}) = (p_{11}, p_{12}, \cdots, p_{1t_1}, p_{21}, p_{22} \cdots, p_{2t_2}, \cdots, p_{m1}, p_{m2}, \cdots, p_{mt_m})$, and $p_{uv} = p_{uw}, u = 1, 2, \cdots, m, v, w \in \{1, 2, \cdots, t_u\}$. The repetition number of these permutations is $\prod_{j=1}^{m} (t_j!)$.

For the divided formula (1) of p, if $\forall p_i, \exists p_j, j \neq j$, and $p_j = p_i$, the element with minimum repetition number can be fixed (The purpose is to simplify the procedure.), and supposed as p_k. Similarly, none of all permutations of $p_1, p_2, \cdots, p_{k-1}$ which are symmetrical has repetition, unless in the following cases only:

(i) since permutation of $ap_2 p_3 \cdots p_{k-1} a$ expresses the same sequence of number (here $p_1 = p_k = a$) as $ap_{k-1} \cdots p_3 p_2 a$, while the permutation of $p_1, p_2, \cdots, p_{k-1}, ap_2 p_3 \cdots p_{k-1}$, is different from that of $ap_{k-1} \cdots p_3 p_2$, unless $p_i = p_{k+1-i}, i = 2, 3, \cdots, k-1$. Therefore it may has a repetition, i. e. when $p_1 = p_k = a$, the inverse of each permutation of $p_2 p_3 \cdots p_{k-1}$ may has a repetition;

(ii) suppose in $p_i (i = 1, 2, \cdots, k)$, the element with minimum repeats appears s times, and $s \geqslant 2$, give $a, p_k = a$ and $(p_{i1}, p_{i2}, \cdots, p_{it_i}) = Q_i, i = 1, 2, \cdots, s, p_{ij} \neq a$, then permutation

$$Q_s a Q_{s-1} a \cdots Q_i a \cdots Q_2 a Q_1 a$$

expresses the same sequence of number as

$$Q_{i-1} a \cdots Q_2 a Q_1 a Q_s a \cdots Q_i a \left(i = 2, 3, \cdots, s, \sum_{i=1}^{s} t_i + s = k \right) .$$

Hence, there is probably a repetition (provided $\exists Q_i \neq Q_j$).

(iii) generally, permutation

$$\underbrace{aa \cdots a}_{q_{l+1}} Q_l \underbrace{aa \cdots a}_{q_l} Q_{l-1} \underbrace{aa \cdots a}_{q_{l+1}} Q_{l-2} \cdots \underbrace{aa \cdots a}_{q_{i+1}} Q_i \cdots \underbrace{aa \cdots a}_{q_2} Q_1 \underbrace{aa \cdots a}_{q_i}$$

expresses the same sequence of number as

$$\underbrace{aa \cdots a}_{q_i - q_0} Q_{i-1} \cdots \underbrace{aa \cdots a}_{q_2} Q_1 \underbrace{aa \cdots a}_{q_{l+1} + q_1} Q_l \cdots \underbrace{aa \cdots a}_{q_{i+1}} Q_i \underbrace{aa \cdots a}_{q_0},$$

$$\left(i = 2, 3, \cdots l, \sum_{i=1}^{s} t_i + s = k, s = \sum_{i=1}^{l+1} q_i, q_i \geqslant 0, q_0 \geqslant 1 \right),$$

and there is probably a repetition (provided $\exists Q_i \neq Q_j$, and $q_i, q_j \geqslant 1$; or $q_i \neq q_j$).

For the divided formula (1) of p, suppose the symmetric number of $p_1, p_2, \cdots, p_{k-1}$ is s_i, the repetition times of (i), (ii) and (iii) are r_i, and the number of corresponding different sequence (2) is

$$d_i = \frac{1}{2} \left(\frac{(k-1)!}{\prod_{j=1}^{m} (t_{ij}!)} + s_i \right) - r_i .$$

Suppose the number of divided formula of p is l, then the distribution-number of k negative edges is

$$f_k = \sum_{i=1}^{l} d_i = \sum_{i=1}^{l} \left[\frac{1}{2} \left(\frac{(k-1)!}{\prod_{j=1}^{m} (t_{ij}!)} + s_i \right) - r_i \right] , \quad (3)$$

and the distribution-number of k negative edges is identical with that of $n - k$ positive edges, therefore the answer of the question is as follows.

Lemma 1 The distribution number of negative edges in a regular n-polygon is

$$s(n)=\sum_{k=0}^{n} f_k=\begin{cases} 2\sum_{k=0}^{\frac{n-1}{2}} f_k, & (n-\text{ odd}), \\ 2\sum_{k=0}^{\frac{n}{2}-1} f_k+f_{\frac{n}{2}}, & (n-\text{ even}). \end{cases}$$

Example 1 Find the number of the homeomorphism class of G. M. .

Solution By Lemma 1, $s(5)=2(f_0+f_1+f_2)$

$f_0=f_1=1$ (by means of f_0 and f_1)

when $k=2, n-k=3=0+3=1+2$,

By Formula (3), $f_2=\frac{1}{2}\left(\frac{1}{1}+1\right)+\frac{1}{2}\left(\frac{1}{1}+1\right)=2$ (here $s_i=1, n=0, i=1,2$).

Hence $s(5)=2(1+1+2)=8$.

Example 2 Find the number of the homeomorphism class of G. M.  11 vertices.

Solution By Lemma 1, $s(10)=2\sum_{k=0}^{4} f_k+f_5$, $f_0=f_1=1$,

when $k=2, n-k=8=0+8=1+7=2+6=3+5=4+4$.

By Formaula (3), $f_2=1+1+1+1+1=5$.

When $k=3$, $n-k=7=(0,0,7)=\underline{(0,1,6)}=\underline{(0,2,5)}=\underline{(0,3,4)}=\underline{(1,1,5)}=\underline{(1,2,4)}=\underline{(1,3,3)}=\underline{(2,2,3)}$.

By Formaula (3), $f_3=1+\frac{2}{2}\times 3+\frac{2}{2}+\left[\frac{1}{2}\left(\frac{2}{2}+1\right)\right]\times 3=8$.

When $k=4$, $n-k=6=\underline{(0,0,1,5)}=\underline{(0,0,2,4)}=\underline{(0,0,3,3)}=\underline{(0,1,1,4)}=\underline{(0,1,2,3)}$
$=\underline{(0,2,2,2)}=\underline{(1,1,1,3)}=\underline{(1,1,2,2)}$.

By Formual (3), $f_4=1+\left[\frac{1}{2}\left(\frac{3}{2}+1\right)\right]\times 5+\frac{3}{2}+\left[\frac{1}{2}\left(\frac{3}{3}+1\right)\right]\times 2=16$.

When $k=5$,

$n-k=5=\underline{(0,0,0,1,4)}=\underline{(0,0,0,2,3)}=\underline{(0,0,1,1,3)}=\underline{(0,0,1,2,2)}=\underline{(0,1,1,1,2)}=\underline{(1,1,1,1,1)}$.

By Formaula (3), $f_5=1+\left(\frac{1}{2}\times\frac{4}{3}\right)\times 2+\left[\frac{1}{2}\times\left(\frac{4}{2\cdot 2}+2\right)\right]\times 2+\frac{1}{2}\times\frac{4}{3}+\frac{1}{2}\times\left(\frac{4}{4}+1\right)=16$.

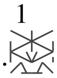
Hence $s(10)=(1+1+5+8+16)\times 2+16=78$.

Example 3 Find the number of the homeomorphism class of G. M. .

Solution By Lemma 1, $s(3)=2\sum_{k=0}^{1} f_k$, but due to its full symmetry, G. M.  is homeomorphic to , the number found is $s(3)-1=3$.

It is easy to see the case of Example 3 only when $n=3$.

Therefore it can be concluded that:

Theorem 1 The number of homeomorphism classes of G. M.  $n+1$ vertices is

$$s(n)=\begin{cases} \sum_{k=0}^{n} f_k=\begin{cases} 2\sum_{k=0}^{\frac{n-1}{2}} f_k, (n\geqslant 5\ \ n-\text{ odd}), \\ 2\sum_{k=0}^{\frac{n}{2}-1} f_k+f_{\frac{n}{2}}, (n\geqslant 4\ n-\text{ even}), \end{cases} \\ \sum_{k=0}^{n} f_k-1 \quad (n=3). \end{cases}$$

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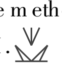
2 Yuan F Y. A simple method for computing the homeomorphism class of G. M.  and G. M. . Chinese Quarterly Journal of Math, 1996, 11(1).

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