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New Lower Bounds for the Ramsey Number $R_4(6)$*
Ramsey数 $R_4(6)$的新下界

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Abstract The 4-colorings of the complete graph K_{929} are investigated by generalizing the work of Mathon. A new lower bound for the Ramsey number $R_4(6)$ is obtained: $R_4(6) \geqslant 3721$.

Key words Ramsey number, lower bound, complete graph

摘要　推广 Mathon的方法,并对 4色完全图 K_{929}进行研究,得到 Ramsey数 $R_4(6)$的新下界: $R_4(6) \geqslant 3721$.

关键词　Ramsey数　下界　完全图
中图法分类号　O157.5; TP312

1 Work of Mathon

In 1987 R. Mathon[1] studied a family of graphs, so-called Paley graphs. Let $q = mt + 1$ be the power of a prime number. Let K_q denote the complete graph with q vertices. Let U be a primitive element of the finite field $GF(q)$ with q elements. Let

$$C_{i+1} = \{U^{mj+i} \mid j = 0, 1, \cdots, t-1\}, \quad (1)$$

for $i = 0, 1, \cdots, m-1$.

A cyclotomic association scheme with m classes corresponds to the edge-coloring of K_q as follows:

$D_m(q)$: an edge (x, y) of K_q has color i if and only if $x - y \in C_i$, $1 \leqslant i \leqslant m$.

Using this construction a graph K_n with $n = m(q+1)$ was defined in [1] and the following lemma played a key role:

Lemma 1 (**Mathon**) Assume that the clique number of $D_m(q)$ is k. Then

$$R_m(k+2) \geqslant m(q+1) + 1.$$

Combined with some known results the above lemma can produce some new lower bounds for Ramsay numbers: It was known that the clique numbers of $D_3(127)$ and $D_3(1069)$ are 3 and 5 respectively. Mathon's lemma implies that $R_3(5) \geqslant 385$ and $R_3(7) \geqslant 3211$; It was known that the clique numbers of $D_4(457)$ and $D_4(857)$ are 3 and 4 respectively. Mathon's lemma implies that $R_4(5) \geqslant 1833$ and $R_4(6) \geqslant 3433$, and so on. These results have been recorded as the best ones in the survey article[3].

2 Generalization of Mathon's lemma

We noticed that the condition "U being a primitive element of $GF(q)$" and the subset $C_1 = \{U^{mj} \mid j = 0, 1, \cdots, t-1\}$ are not essential. They can be relaxed in the following way.

Let $F_q^* = GF(q) \backslash \{0\}$. Assume that there exists $T \in F_q^*$ and a parameter set $C_0 \subset F_q^*$ such that

$$C_i = T^i C_0, i = 0, 1, \cdots, m-1, \quad (2)$$

form an m-partition of F_q^*. Define $D_m(q)$, $(0 \leqslant i \leqslant m-1)$. as in (1). The proofs of Lemma 1 and Theorem 2 in references [1] are still valid when U is replaced by T. Hence Lemma 1 still holds under our new assumption. This generalizes Mathon's lemma.

3 A circulant graph of prime order

Based on our work in references [2~9] we found

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C_0 and T as described in the previous section. Let p be the prime number 929 and let S_0 be the set

{1, 9, 12, 20, 30, 32, 35, 38, 44, 46, 49, 50, 51, 56, 59, 65, 68, 71, 72, 73, 77, 82, 91, 97, 99, 104, 108, 116, 122, 126, 143, 148, 155, 163, 169, 170, 173, 180, 183, 188, 191, 201, 203, 205, 206, 210, 212, 215, 217, 218, 222, 229, 234, 240, 243, 248, 251, 256, 259, 261, 267, 281, 285, 289, 301, 302, 304, 305, 313, 315, 316, 317, 318, 329, 330, 336, 341, 344, 345, 347, 352, 354, 361, 368, 370, 371, 375, 377, 379, 381, 384, 390, 392, 394, 397, 398, 399, 400, 401, 403, 404, 405, 408, 418, 421, 423, 431, 437, 438, 445, 446, 448, 454, 456, 462, 463}.

We use the method in references [8] in the following computation of the clique number of the circulant graph $G_p(S_0)$, i. e., the number $D_m(q)$ with $q = p, m = 4$ in the previous section. Let

$$A_0 = \{x: |x| \in S_0\},$$

$$d(x) = |\{y \in A_0: |y - x| \in S_0\}|.$$

The computation shows that $d(1) = d(20) = d(72) = d(99) = \cdots = 48$, $d(12) = d(35) = d(59) = d(65) = \cdots = 50, \cdots$. Let $(A_0, \prec)$ be the totally ordered set

{1, − 1, 20, − 20, 72, − 72, 99, − 99, 122, − 122, 148, − 148, 173, − 173, 201, − 201, 210, − 210, 212, − 212, 256, − 256, 261, − 261, 304, − 304, 347, − 347, 352, − 352, 354, − 354, 361, − 361, 368, − 368, 379, − 379, 390, − 390, 392, − 392, 400, − 400, 405, − 405, 408, − 408, 418, − 418, 423, − 423, 437, − 437, 445, − 445, 454, − 454, 12, − 12, 35, − 35, 59, − 59, 65, − 65, 68, − 68, 82, − 82, 97, − 97, 126, − 126, 155, − 155, 215, − 215, 218, − 218, 229, − 229, 234, − 234, 240, − 240, 243, − 243, 251, − 251, 259, − 259, 267, − 267, 285, − 285, 313, − 313, 330, − 330, 345, − 345, 371, − 371, 375, − 375, 394, − 394, 397, − 397, 421, − 421, 431, − 431, 448, − 448, 9, − 9, 30, − 30, 32, − 32, 38, − 38, 44, − 44, 46, − 46, 49, − 49, 50, − 50, 51, − 51, 56, − 56, 71, − 71, 73, − 73, 77, − 77, 91, − 91, 104, − 104, 108, − 108, 116, − 116, 143, − 143, 163, − 163, 169, − 169, 170, − 170, 180, − 180, 183, − 183, 188, − 188, 191, − 191, 203, − 203, 205, − 205, 206, − 206, 217, − 217, 222, − 222, 248, − 248, 281, − 281, 289, − 289, 301, − 301, 302, − 302, 305, − 305, 315, − 315, 316, − 316, 317, − 317, 318, − 318, 329, − 329, 336, − 336, 341, − 341, 344, − 344, 370, − 370, 377, − 377, 381, − 381, 384, − 384, 398, − 398, 399, − 399, 401, − 401, 403, − 403, 404, − 404, 438, − 438, 446, − 446, 456, − 456, 462, − 462, 463, − 463}.

In order to compute the S_0-colored chains starting with $a = 1$ we set

$\{y \in A_0: |y - 1| \in S_0\}$ = {72, − 72, − 304, 400, − 400, 405, − 437, − 445, 218, 330, 345, 371, − 397, − 49, 50, − 50, 51, − 71, 73, − 169, 170, − 205, 206, − 217, − 301, 302, 305, − 315, 316, − 316, 317, − 317, 318, − 329, − 344, − 370, 398, − 398, 399, − 399, 401, − 403, 404, − 404, 438, 446, − 462, 463}.

The computation shows that the first S_0-chain of length 2 starting with 1 is $1 \prec 72 \prec -50$ and there are no longer chains. Hence $l(1) = 2$.

Similar computation shows that $l(a) \leqslant 2$ for any $a \in S_0$. By Lemma 2 and Lemma 3 in refdrences [8] the clique number of $G_p(S_0)$ is $k = c(G_p(S_0)) = 2 + \max\{l(a) | a \in S_0\} = 4$.

4 A new lower bound for $R_4(6)$

Let $T = 4$ and $S_i = \{T^i x | x \in S_0\}$. By computation, we obtain

S_1 = {4, 6, 7, 10, 13, 31, 34, 36, 41, 42, 43, 48, 57, 61, 63, 66, 69, 74, 75, 78, 80, 81, 89, 95, 105, 106, 107, 109, 110, 115, 117, 120, 125, 128, 134, 139, 140, 152, 165, 166, 174, 176, 177, 184, 186, 195, 196, 197, 200, 204, 209, 211, 224, 226, 227, 236, 237, 238, 242, 246, 249, 253, 254, 258, 260, 262, 266, 270, 272, 275, 277, 279, 282, 284, 287, 288, 290, 291, 292, 298, 308, 309, 322, 323, 328, 331, 334, 335, 337, 339, 342, 343, 350, 357, 358, 364, 374, 378, 386, 387, 388, 391, 396, 414, 415, 416, 425, 432, 435, 441, 442, 447, 450, 451, 459, 464},

S_2 = {2, 15, 16, 19, 21, 22, 23, 24, 25, 28, 33, 39, 40, 52, 54, 55, 58, 67, 70, 83, 85, 87, 90, 93, 94, 103, 111, 113, 118, 119, 124, 129, 130, 135, 136, 141, 144, 145, 149, 151, 158, 159, 164, 168, 171, 172, 179, 185, 187, 192, 193, 194, 198, 199, 202, 207, 219, 221, 223, 225, 228, 231, 233, 235, 239, 244, 252, 263, 264, 265, 269, 274, 276, 294, 296, 300, 303, 306, 307, 310, 312, 314, 320, 321, 324, 346, 356, 359, 362, 363, 369, 373, 380, 383, 393, 395, 402, 407, 411, 417, 419, 420, 424, 426, 427, 428, 429, 430, 436, 439, 440, 443, 449, 458, 460, 461},

$S_3 = \{3, 5, 8, 11, 14, 17, 18, 26, 27, 29, 37, 45, 47, 53, 60, 62, 64, 76, 79, 84, 86, 88, 92, 96, 98, 100, 101, 102, 112, 114, 121, 123, 127, 131, 132, 133, 137, 138, 142, 146, 147, 150, 153, 154, 156, 157, 160, 161, 162, 167, 175, 178, 181, 182, 189, 190, 208, 213, 214, 216, 220, 230, 232, 241, 245, 247, 250, 255, 257, 268, 271, 273, 278, 280, 283, 286, 293, 295, 297, 299, 311, 319, 325, 326, 327, 332, 333, 338, 340, 348, 349, 351, 353, 355, 360, 365, 366, 367, 372, 376, 382, 385, 389, 406, 409, 410, 412, 413, 422, 433, 434, 444, 452, 453, 455, 457\}$.

Let $C_i = \{x: |x| \in S_i\}$ for $i = 0, 1, 2, 3$. Then $\{C_0, C_1, C_2, C_3\}$ is a partition of F_q^*. Evidently $G_p(C_i)$ and $G_p(C_0)$ (i.e. $G_p(S_i)$ and $G_p(S_0)$) are isomorphic. Since $G_p(S_0)$ is just $D_m(q)$ with $q = p = 929, m = 4$ as explained before and the clique number of $D_m(q)$ is equal to 4, by Mathon's lemma we obtain the following result:

Theorem 1 $R_4(6) \geqslant 3721$.

This result surpasses the record $R_4(6) \geqslant 3433$ in reference [10].

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