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On the Existnece and Uniqueness of Almost Periodic Solutions of Some Delay Differential Equations*
一类时滞微分方程概周期解的存在唯一性

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Abstract By means of Liapunov functional, the existence and uniqueness of almost periodic solutions for some delay differential equations was investigated.

Key words delay differential equation, almost periodic solution, existence and uniqueness

摘要　应用 Liapunov 泛函, 研究一类时滞微分方程概周期解的存在唯一性.

关键词　时滞微分方程　概周期解　存在唯一性
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Consider the following delay differential equation

$$x''(t)+bx'(t)+f(x(t))+g(x(t-\tau))=e(t), \tag{1}$$

where f and g are continuous functions, $e(t)$ is a real continuous almost periodic function, b is a positive constant and delay $\tau>0$.

It is known that second order differential equations appear frequently in various engineering technology problems. Many authors have studied the qualitative properties for these equations, such as stability, boundedness and existence of periodic solutions[18]. Theoretically, one can investigate the existence and uniqueness of almost periodic solutions for differential equations by Liapunov function or Liapunov functional[912]. However, how to construct such Liapunov function or Liapunov functional for the specific system is still a problem.

In this paper, we discuss the existence and uniqueness of almost periodic solutions for system (1) by constructing Liapunov functional. A sufficient condition on the existence and uniqueness of almost periodic solutions for equation (1) is obtained.

Definition 1 $f(t,\phi)$ is said to be uniformly almost periodic with respect to t for $\phi\in\Omega$, if for any $\varepsilon>0$ and compact set $W(W\in\Omega)$, there exists an $l=l(\varepsilon,W)>0$ such that $|f(t+\tau,\phi)-f(t,\phi)|<\varepsilon$, $t\in R$, $\phi\in W$ and for any region of length l containing a τ.

For equation (1), let $G(x)=\int_0^x[f(u)+g(u)]\,du$, $E(t)=\int_0^t e(s)ds$. Suppose that $E(t)$ is bounded, then $E(t)$ is also an almost periodic function[9]. Let $M=\sup_{t\in R}|E(t)|$, then we have

Theorem 1 Assume that the following conditions are satisfied.

(Ⅰ) $f(x)$ and $g(x)$ both are continuously differentiable functions for $|x|<+\infty$; there is a positive constant c such that $0<g'(x)\leqslant c$ and $G(x)\to+\infty(|x|\to+\infty)$.

(Ⅱ) There exists a positive constant K such that $f(x)+g(x)>0(x\geqslant K)$; $f(x)+g(x)<0(x\leqslant -K)$, and, $f(x)$ and $g(x)$ satisfy

$$\begin{cases} bx[f(x)+g(x)]-M[f(x)+g(x)]-\\ \quad\frac{\tau}{2}[f(x)+g(x)]^2-\frac{\tau}{2}g^2(x)>0,\ (x\geqslant K)\\ bx[f(x)+g(x)]+M[f(x)+g(x)]-\\ \quad\frac{\tau}{2}[f(x)+g(x)]^2-\frac{\tau}{2}g^2(x)>0,\ (x\leqslant -K)\end{cases} \tag{2}$$

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then there exists a bounded solution of equation (1).

Proof Consider the equivalent system of equation (1) as follows:

$$\begin{cases} x' = y - bx + E(t) + \int_{t-\tau}^{t} g(x(s))ds, \\ y' = -f(x) - g(x), \end{cases} \quad (3)$$

where $x = x(t), y = y(t)$. Let $(x, y) = (x(t), y(t))$ be a solution of system (3), consider a Liapunov functional

$$W(t) = \frac{1}{2}y^2 + G(x) + \frac{1}{2}\int_{-\tau}^{0}\int_{t+s}^{t} g^2(x(\theta))d\theta ds, \quad (4)$$

then we have

$$W'_{(3)}(t) = y[-f(x) - g(x)] + [f(x) + g(x)][y - bx + E(t) + \int_{t-\tau}^{t} g(x(s))ds] + \frac{1}{2}\int_{-\tau}^{0} g^2(x(t))ds - \frac{1}{2}\int_{-\tau}^{0} g^2(x(t+s))ds = -bx[f(x) + g(x)] + E(t)[f(x) + g(x)] + [f(x) + g(x)]\int_{t-\tau}^{t} g(x(s))ds + \frac{\tau}{2}g^2(x) - \frac{1}{2}\int_{t-\tau}^{t} g^2(x(s))ds. \quad (5)$$

Notice that

$$[f(x) + g(x)]\int_{t-\tau}^{t} g(x(s))ds = \int_{t-\tau}^{t}[f(x(t)) + g(x(t))]g(x(s))ds \leqslant \frac{1}{2}\int_{t-\tau}^{t}[f(x(t)) + g(x(t))]^2 ds + \frac{1}{2}\int_{t-\tau}^{t} g^2(x(s))ds = \frac{1}{2}[f(x(t)) + g(x(t))]^2\int_{t-\tau}^{t} ds + \frac{1}{2}\int_{t-\tau}^{t} g^2(x(s))ds = \frac{\tau}{2}[f(x) + g(x)]^2 + \frac{1}{2}\int_{t-\tau}^{t} g^2(x(s))ds. \quad (6)$$

So we get

$$W'_{(3)}(t) \leqslant -bx[f(x) + g(x)] + E(t)[f(x) + g(x)] + \frac{\tau}{2}[f(x) + g(x)]^2 + \frac{\tau}{2}g^2(x). \quad (7)$$

From condition (Ⅱ), when $x \geqslant K$, we have $f(x) + g(x) > 0$, therefore,

$$W'_{(3)}(t) \leqslant -bx[f(x) + g(x)] + M[f(x) + g(x)] + \frac{\tau}{2}[f(x) + g(x)]^2 + \frac{\tau}{2}g^2(x) < 0, \quad (8)$$

when $x < -K$, we have $f(x) + g(x) < 0$, and

$$W'_{(3)}(t) \leqslant -bx[f(x) + g(x)] - M[f(x) + g(x)] + \frac{\tau}{2}[f(x) + g(x)]^2 + \frac{\tau}{2}g^2(x) < 0. \quad (9)$$

From the definition of $W(t)$, we know that $W(t) \to +\infty$ when $|x| \to +\infty$ and $|y| \to +\infty$, and from $W'_{(3)}(t) < 0$ ($|x| \geqslant K$), we know that (x, y) is bounded.

In order to study the existence and uniqueness of almost periodic solutions of equation (1), consider the product system of (3) as follows:

$$\begin{cases} x' = y - bx + E(t) + \int_{t-\tau}^{t} g(x(s))ds, \\ y' = -f(x) - g(x), \\ u' = v - bu + E(t) + \int_{t-\tau}^{t} g(u(s))ds, \\ v' = -f(u) - g(u). \end{cases} \quad (10)$$

Let $f_1(t) = f(x(t),$

$$u(t)) = \begin{cases} \frac{f(x) - f(u)}{x - u}, x \neq u; \\ f'(x), x = u. \end{cases}$$

$$g_1(t) = g(x(t), u(t)) = \begin{cases} \frac{g(x) - g(u)}{x - u}, x \neq u; \\ g'(x), x = u. \end{cases}.$$

$X = X(t) = x(t) - u(t)$, $Y = Y(t) = y(t) - v(t)$; $A = 2b - \frac{1}{2}f_1(t) - \frac{1}{2}g_1(t) - \frac{5\tau}{4}g_1^2(t) - \tau$, $B = 2 + \frac{b}{2} - 4f_1(t) - 4g_1(t)$, $C = \frac{1}{2} - \frac{\tau}{4}$.

Suppose that

(Ⅲ) $A > 0$, $C > 0$, $B^2 < 4AC$.

Theorem 2 Assume that conditions (Ⅰ)(Ⅲ) hold, then system (3), namely, the equation (1) has one and only one almost periodic solution.

Proof From Theorem 1, there exists a bounded solution of system (1). For the product system (10), consider a Liapunov functional as follows:

$$V(t) = X^2 - \frac{1}{2}XY + 2Y^2 + \frac{5}{4}\int_{-\tau}^{0}\int_{t+s}^{t} g_1^2(\theta)X^2(\theta)d\theta ds. \quad (11)$$

Then we have

$$V'_{(10)}(t) = 2X\{Y - bX + \int_{t-\tau}^{t}[g(x(s)) - g(u(s))]ds\} - \frac{1}{2}Y\{Y - bX + \int_{t-\tau}^{t}[g(x(s)) - g(u(s))]ds\} + \frac{1}{2}X\{[f(x) - f(u)] + [g(x) - g(u)]\} - 4Y\{[f(x) - f(u)] + [g(x) - g(u)]\} + \frac{5}{4}\int_{-\tau}^{0} g_1^2(t)X^2(t)ds - \frac{5}{4}\int_{-\tau}^{0} g_1^2(t+s)X^2(t+s)ds = 2XY - 2bX^2 + 2X\int_{t-\tau}^{t} g_1(s)X(s)ds - \frac{1}{2}Y^2 +$$

$\frac{b}{2}XY - \frac{1}{2}Y\int_{t-\tau}^{t} g_1(s)X(s)\mathrm{d}s + [\frac{1}{2}f_1(t) + \frac{1}{2}g_1(t)]X^2 - [4f_1(t) + 4g_1(t)]XY + \frac{5\tau}{4}g_1^2(t)X^2 - \frac{5}{4}\int_{t-\tau}^{t} g_1^2(s)X^2(s)\mathrm{d}s.$ (12)

Since

$$\begin{cases} 2X\int_{t-\tau}^{t} g_1(s)X(s)\mathrm{d}s \leqslant \tau X^2 + \int_{t-\tau}^{t} g_1^2(s)X^2(s)\mathrm{d}s, \\ -\frac{1}{2}Y\int_{t-\tau}^{t} g_1(s)X(s)\mathrm{d}s \leqslant \frac{\tau}{4}Y^2 + \frac{1}{4}\int_{t-\tau}^{t} g_1^2(s)X^2(s)\mathrm{d}s, \end{cases} \quad (13)$$

thus we get

$V'_{(10)}(t) \leqslant -(2b - \frac{1}{2}f_1(t) - \frac{1}{2}g_1(t) - \frac{5\tau}{4}g_1^2(t) - \tau)X^2 + (2 + \frac{b}{2} - 4f_1(t) - 4g_1(t))XY - (\frac{1}{2} - \frac{\tau}{4})Y^2 = -AX^2 + BXY - CY^2 = -C(Y - \frac{B}{2C}X)^2 - A(1 - \frac{B^2}{4AC})X^2.$ (14)

Again we have

$V'_{(10)}(t) \leqslant -AX^2 + BXY - CY^2 = -A(X - \frac{B}{2A}Y)^2 - C(1 - \frac{B^2}{4AC})Y^2.$ (15)

Combined with (14), we obtain

$V'_{(10)}(t) \leqslant -\frac{1}{2}A(1 - \frac{B^2}{4AC})X^2 - \frac{1}{2}C(1 - \frac{B^2}{4AC})Y^2.$ (16)

From condition (III) we know that there exists a positive constant α such that

$V'_{(10)}(t) \leqslant -\alpha(X^2 + Y^2).$ (17)

According to Yoshizawa's theorem of the existence and uniqueness of almost perodic solutions[9,12], we know that there exists one and only one almost perodic solution of equation (1).

Example Comsider the following system

$x''(t) + 2x'(t) + \frac{x}{8(1+x^2)} + \frac{1}{2}x(t - \frac{1}{10}) = e(t),$ (18)

where $b = 2$, $f(x) = \frac{x}{8(1+x^2)}$, $g(x) = \frac{1}{2}x$. $\tau = \frac{1}{10}$, $e(t)$ is a real continuous almost periodic function.

Let $M = \sup_{t\in R}|E(t)|$, and $K = 2M + 1$, then

$bx[f(x) + g(x)] - M|f(x) + g(x)| - \frac{\tau}{2}[f(x) + g(x)]^2 - \frac{\tau}{2}g^2(x) = 2x(\frac{x}{8(1+x^2)} + \frac{x}{2}) - M|\frac{x}{8(1+x^2)} + \frac{x}{2}| - \frac{1}{20}(\frac{x}{8(1+x^2)} + \frac{x}{2})^2 - \frac{1}{20}(\frac{x}{2})^2 > \frac{x^2}{2} - \frac{3M|x|}{4} > 0, (|x| > K),$ (19)

$-\frac{1}{64} \leqslant f_1(t) \leqslant \frac{1}{8}$, $g_1(t) = \frac{1}{2}$;

$A \geqslant 4 - \frac{1}{2}\times\frac{1}{8} - \frac{1}{2}\times\frac{1}{2} - \frac{5}{4}\times\frac{1}{10}\times(\frac{1}{2})^2 - \frac{1}{10} = 4 - \frac{1}{16} - \frac{1}{4} - \frac{5}{160} - \frac{1}{10} > 3\frac{1}{2}$;

$|B| \leqslant 2 + 1 + 4\times\frac{1}{8} - 4\times\frac{1}{2} = 1\frac{1}{2}$;

$C = \frac{1}{2} - \frac{1}{4}\times\frac{1}{10} = \frac{19}{40}$.

We know that $B^2 < 4AC$ is valid. From Theorem 2, there exists one and only one almost periodic solution of equation (18).

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