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Characteristics of Open Subtrees Without Periodic Points of A Tree Map *

树映射中不含周期点的开子树的特征

Zhang Yongping Zhang Xiaoyan Xu Shengrong Sun Taixiang
张永平 张晓燕 徐胜荣 孙太祥

(Coll. of Math. & Info. Sci., Guangxi Univ., 100 Daxuelu, Nanning, Guangxi, 530004, China)
(广西大学数学与信息科学学院 南宁市大学路 100 号 530004)

Abstract Let T be a tree and f be a continuous map from T into itself. Some properties of open subtrees of T without periodic points of f are discussed.

Key words tree map, ω- limit set, open subtree, recurrent point, non-wandering set

摘要 讨论 T 是树且 f 是 T 的连续自映射时, T 中不含 f 周期点的开子树的一些性质.

关键词 树映射 开子树 ω- 极限集 回归点 非游荡集
中图法分类号 O189.11

1 Introduction

In this paper, let N be the set of all natural numbers. Write $Z^+ = N \cup \{0\}$, $N_n = \{1, 2, \cdots, n\}$ and $Z_n = \{0\} \cup N_n$ for any $n \in N$.

Let T be a tree (i.e. an one-dimensional compact connected branched manifold without cycles). A subtree of T is a subset of T, which is a tree itself. For any $x \in T$, denote by $V(x)$ the number of connected components of $T - \{x\}$. $B(T) = \{x \in T: V(x) \geqslant 3\}$ is called the set of branched points of T and $E(T) = \{x \in T: V(x) = 1\}$ is called the set of end points of T. Let $NE(T)$ be the number of end points of T. Let $A \subset T$, we use $\bar{A}$, $\dot{A}$, $[A]$ and $\#(A)$ to denote the closure of A, the interior of A, the smallest subtree of T containing A and the number of points in A respectively. For any $x, y \in T$, we shall use $[x, y]$ to denote $[\{x, y\}]$. Define $(x, y] = [x, y] - \{x\}$ and $(x, y) = (x, y] - \{y\}$. For any $x \in T$ and any $\varepsilon > 0$, write $B(x, \varepsilon) = \{y \in T: d(x, y) < \varepsilon\}$ and $B_1(x, \varepsilon)$, $B_2(x, \varepsilon)$, $\cdots$, $B_{V(x)}(x, \varepsilon)$ be connected components of $B(x, \varepsilon) - \{x\}$.

Let $C^0(T)$ be the set of all continuous maps from T to itself. For any $f \in C^0(T)$ and any $x \in T$, the set of fixed points of f, the set of m- periodic points of f, the ω- limit set of x, the set of non-wandering points of f will be denoted by $F(f)$, $P_m(f)$, $\omega(x, f)$, $\Omega(f)$ respectively. Write $O(x, f) = \{f^k(x): k \in Z^+\}$ and $P(f) = \bigcup_{m=1}^{\infty} P_m(f)$.

Block and Coven in Reference [1] studied some properties of open subintervals of $[0, 1]$ without periodic points of $f \in C^0([0, 1])$ and obtained the following theorem.

Theorem A Let $f \in C^0([0, 1])$.

(1) If $x \in \Omega(f) - \overline{P(f)}$, then there exists a $\delta > 0$ such that, for any $\varepsilon \in (0, \delta)$ we have $J \cap f^n(J_1) = \varphi$ for all $n > 0$, where $J = [x - \varepsilon, x + \varepsilon]$ and J_1 denotes exactly one of $[x, x + \varepsilon]$, $[x - \varepsilon, x]$.

(2) Let J be an open subinterval of $[0, 1]$ which contains no periodic point of f. Then

(i) J contains at most one point of any limit set $\omega(x)$.

(ii) if $x \in J$ is non-wandering, then no other point of its trajectory lies in J.

In this note, we extend Theorem A to a tree map and obtain the following two theorems.

Theorem 1 Let $f \in C^0(T)$ and $m = V(x)$. If $x \in \Omega(f) - \overline{P(f)}$, then there exist a $\delta > 0$ and $j \in N_m$ such that

$$B(x, \delta) \cap f^n(B_j(x, \delta)) = \varphi \text{ for all } n \in N.$$

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Theorem 2 Let $f \in C^0(T)$, $U \subset T - \overline{P(f)}$ be an open subtree and $s = NE(U)$. Then

(1) for any $x \in T$, $\#(U \cap \omega(x,f)) \leqslant s-1$.

(2) for any $x \in \Omega(f)$, $\#(U \cap O(x,f)) \leqslant s-1$.

2 Some Lemmas

To prove the main theorems, we first give some lemmas.

Lemma 1[1] Let $f \in C^0(T)$. If there exist $x, y \in T$ such that $[x, y] \subset [f(x), f(y)]$, then $[x, y] \cap F(f) \neq \varphi$

Lemma 2 Let $f \in C^0(T)$ and $U \subset T - \overline{P(f)}$ be an open subtree. Suppose $x \in U$ such that $f^n(x) \in U$ for some $m \in N$. If V is a connected component of $U - \{x\}$ containing no $f^m(x)$, then $V \cap O(x, f) = \varphi$.

Proof We first show that for any $k \in N$, $f^{km}(x)$ belongs to the connected component W of $T - \{x\}$ containing $f^m(x)$.

Assume on the contrary that for some $s \in N$, $f^{sm}(x) \notin W$. Let $k = \min\{s: f^{sm}(x) \notin W\}$. Then we have

$$x \in [f^{(k-1)m}(x), f^{km}(x)] \subset f^{(k-1)m}[x, f^m(x)].$$

So there exists a point $\zeta \in [x, f^m(x)]$ such that $f^{km}(\zeta) = f^m(x)$. By Lemma 1, it follows that $[x, \zeta] \cap F(f^{km}) \neq \varphi$. This is a contradiction.

Now we show $V \cap O(x, f) = \varphi$.

Assume on the contrary that $V \cap O(x, f) \neq \varphi$. Then there exists a $n \in N$ such that $f^n(x) \in V$. From the above we know that $f^{kn}(x) \in V$ for any $k \in N$. Therefore, $f^{mn}(x) \in W \cap V = \varphi$. This is a contradiction.

Lemma 3[2] Let $f \in C^0(T)$ and U be a subtree. Let $x_1, x_2, \cdots, x_m$ be m boundary points of U. If $[x_i, f(x_i)] \cap U \neq \varphi$ for each $i \in N_m$, then $U \cap F(f) \neq \varphi$.

3 The Proof of the Main Theorems

Proof of Theorem 1 Since $x \in \Omega(f) - \overline{P(f)}$, we can take $\varepsilon_0 > 0$ such that $B(x, \varepsilon_0) \cap P(f) = \varphi$ and $B(x, \varepsilon_0) \cap B(T) \subset \{x\}$. By Lemma 2.1 in Reference [3], it follows that there exist points $x_k \to x$ in T and natural numbers $n_k \to \infty$ such that $f^{n_k}(x_k) = x$ for all $k \in N$. Without loss of generality, we can suppose $x_k \in B(x, \varepsilon_0)$ for all $k \in N$.

Claim 1 There at least exists a $i \in N_m$, such that $\{x_k\} \cap B_i(x, \varepsilon_0) = \varphi$.

Assume on the contrary that for each $i \in N_m$, $\{x_k\} \cap B_i(x, \varepsilon_0) \neq \varphi$. Let $y_i \in \{x_k\} \cap B_i(x, \varepsilon_0)$ and $f^{k_i}(y_i) = x$ for some $k_i \in N$. By Lemma 2, there exist some $n \in N$ such that $[f^n(y_i), y_i] \cap [y_i, x] \neq \varphi$ for each $i \in N_m$, it follows from Lemma 3 that $B(x, \varepsilon_0) \cap P(f) \neq \varphi$. This is a contradiction.

Without loss of generality, we may suppose that $B_i(x, \varepsilon_0) \cap \{x_k\} \neq \varphi$ for $1 \leqslant i \leqslant l$ and $B_i(x, \varepsilon_0) \cap \{x_k\} = \varphi$ for $l+1 \leqslant i \leqslant m$. It follows from Lemma 2 and Lemma 3 that there exist some $l+1 \leqslant \lambda \leqslant m$ such that $B_j(x, \varepsilon_0) \cap f^n(B_\lambda(x, \varepsilon_0)) = \varphi$ for all $n \in N$ and each $j \in N_m - \{\lambda\}$. We may suppose that if $l+1 \leqslant h \leqslant \lambda \leqslant m$, then $B_j(x, \varepsilon_0) \cap f^n(B_\lambda(x, \varepsilon_0)) = \varphi$ for all $n \in N$ and each $j \in N_m - \{\lambda\}$.

Claim 2 There exist some $h \leqslant \lambda \leqslant m$ and a $\varepsilon_1 \in (0, \varepsilon_0)$ such that

$$f^n(B_\lambda(x, \varepsilon_1)) \cap B_\lambda(x, \varepsilon_1) = \varphi \text{ for all } n \in N.$$

Assume on the contrary that for each $h \leqslant j \leqslant m$ and any $\varepsilon \in (0, \varepsilon_0)$,

$$f^{m_j}(B_j(x, \varepsilon_0)) \cap B_j(x, \varepsilon_0) \neq \varphi \text{ for some } m_j \in N.$$

Then, by Proposition IV. 6 in Reference [1] and the remark following its proof, for each $h \leqslant j \leqslant m$, there exist a point $y_j \in T$ and a sequence of integers $m_k^j \to \infty$ such that $f^{m_{k+1}^j}(y_j) \in (x, f^{m_k^j}(y_j))$ for all $k \in N$ and $f^{m_k^j}(y_j) \to x$ as $k \to \infty$. Thus, it follows from Lemma 2 and Lemma 3 that $B(x, \varepsilon_0) \cap P(f) \neq \varphi$. This is a contradiction.

Take $\delta = \varepsilon_1$, by Claim 2, we have that

$$f^n(B_\lambda(x, \delta)) \cap B(x, \delta) = \varphi \text{ for all } n \in N.$$

Proof of Theorem 2 (1) Put $\#(U \cap \omega(x, f)) = k$, then $k \neq \infty$. Otherwise there exists some component of $U - B(T)$ containing infinite points of $\omega(x, f)$, which is impossible. Therefore there exist k pairwise disjoint open connected subsets, denoted by $U_1, U_2, \cdots, U_k$, such that every $U_i (i \in N_k)$ is contained in one of the connected components of $U - \{B(T) \cup E(T)\}$ and every $U_i (i \in N_k)$ contains infinite points of $O(x, f)$. Now we prove (1) of Theorem 2 by induction.

(i) If $s = 2$, it is clear that $k \leqslant 1$.

(ii) Assume that (1) of Theorem 2 holds for $2 \leqslant s \leqslant m$, that is to say $k \leqslant s-1$. Now we show that (1) of Theorem 2 holds for $s = m+1$.

Let $y_1, y_2, \cdots, y_s$ be the end points of U and z_i be the nearest branched point to y_i for each $i \in N_s$.

Claim 3 There must exists a $i \in N_s$ such that $(y_i, z_i) \cap (\bigcup_{j=1}^{k} U_j) = \varphi$.

Assume on the contrary that for each $j \in N_s$, we have some $U_j \subset (y_j, z_j)$. For any $f^l(x), f^\lambda(x) \in U_j$ with $l, \lambda \in N$, if $l < \lambda$, then $f^l(x) \in (f^\lambda(x), y_j)$. In fact, if $f^\lambda(x) \in (f^l(x), y_j)$, put $a = f^l(x)$, then $f^\lambda(x) = f^{\lambda - l}(a)$. By Lemma 2, we know that $\bigcup_{i \neq j} U_i \cap O(x, f) = \varphi$, which is a contradiction.

For each $i \in N_s$, choose $\lambda_i > l_i$ with $\lambda_i, l_i \in N$ and $f^{\lambda_i}(x), f^{l_i}(x) \in U_i$. Since $f^{l_i}(x) \in (f^{\lambda_i}(x), y_i)$, by Lemma 2, we have

$$[f^{k(\lambda_i - l_i)}(f^{l_i}(x)), f^{l_i}(x)] \cap [f^{l_i}(x), f^{\lambda_i}(x)] \neq \varphi$$

for all $k \in N$.

However, it follows from Lemma 3 that $U \cap P(f) \neq \varphi$, which contradicts to $U \cap P(f) = \varphi$.

Without loss of generality, we may assume that $(y_1, z_1) \cap (\bigcup_{j=1}^{k} U_j) = \varphi$, $X_1 = (y_1, z_1), X_2, \cdots, X_l$ are l connected components of $U - \{z_1\}$ and $k_i = NE(X_i)(i \in N_l)$. Then we have

$$1 + k_2 - 1 + k_3 - 1 + \cdots + k_l - 1 = s$$

and

$$k_i \leqslant s - 1, i \in (2, 3, \cdots, l\}.$$

For each $i \in \{2, 3, \cdots, l\}$, let s_i be the number of U_j in X_i. By the inductive hypothesis, we know $s_i \leqslant k_i - 1$. Therefore

$$k = s_2 + s_3 + \cdots + s_l \leqslant k_2 - 1 + k_3 - 1 + \cdots + k_l - 1 = s - 1 = m.$$

This completes the proof of (1) of Theorem.

(2) For any $x \in \Omega(f)$, there exist points $x_k \to x$ and integers $n_k \to \infty$ such that $f^{n_k}(x_k) = x$. It is clear that for any $i \in N$, we have

$$f^i(x_k) \to f^i(x) \text{ and } f^{n_k}(f^i(x_k)) = f^i(x).$$

Put $\#(U \cap O(x, f)) = r$, then $r \neq \infty$. Otherwise there exists some component of $U - B(T)$ containing infinite points of $O(x, f)$, which is impossible. Let $f^{m_1}(x), f^{m_2}(x), \cdots, f^{m_r}(x)$ be r points of $U \cap O(x, f)$. Thus there exist r pairwise disjoint open connected subsets, denoted by $U_1, U_2, \cdots, U_r$, such that every U_i whose an end is $f^{m_i}(x)(i \in N_r)$ is contained in one of the connected components of $U - \{B(T) \cup E(T)\}$ and every $U_i (i \in N_r)$ contains infinite points of $\{f^{m_i}(x_k)\}$. By taking a subsequence, we may assume that for each $i \in N_r$ and each $k \in N$, $f^{n_i}(x_{k+1}) \in (f^{m_i}(x), f^{n_i}(x_k)) \subset U_i$.

Now we will prove (2) of Theorem 2 by induction.

(i) If $s = 2$, it is clear that $r \leqslant 1$.

(ii) Assume that (2) of Theorem 2 holds for $2 \leqslant s \leqslant m$, that is to say $r \leqslant s - 1$. Now we show that (2) of Theorem holds for $s = m + 1$.

Let $y_1, y_2, \cdots, y_s$ be the end points of U and z_i be the nearest branched point to y_i for each $i \in N_s$.

Claim 4 There must exists a $i \in N_s$ such that $(y_i, z_i) \cap (\bigcup_{j=1}^{r} U_j) = \varphi$.

Assume on the contrary that for each $j \in N_s$, we have some $U_j \subset (y_j, z_j)$. By Lemma 3, we have that $f^{m_j}(x_k) \in (f^{m_j}(x), y_j)$ and $(f^{m_j}(x_k), f^{m_j}(x)) \cap (f^{m_j}(x_k), f^{sm_j}(x_k)) \neq \varphi$ for each $s, k \in N$ and each $i \in N_s$. Thus, it follows from Lemma 3 that $U \cap P(f) \neq \varphi$. This is a contradiction.

Without loss of generality, we may assume that $(y_1, z_1) \cap (\bigcup_{j=1}^{r} U_j) = \varphi$, $X_1 = (y_1, z_1), X_2, \cdots, X_l$ are l connected components of $U - \{z_1\}$ and $k_i = NE(X_i)(i \in N_l)$. Then we have

$$1 + k_2 - 1 + k_3 - 1 + \cdots + k_l - 1 = s$$

and

$$k_i \leqslant s - 1, i \in \{2, 3, \cdots, l\}.$$

For each $i \in \{2, 3, \cdots, l\}$, let s_i be the number of U_j in X_i. By the inductive hypothesis, we know $s_i \leqslant k_i - 1$. Therefore

$$r = s_2 + s_3 + \cdots + s_l \leqslant k_2 - 1 + k_3 - 1 + \cdots + k_l - 1 = s - 1 = m.$$

This completes the proof of (2) of Theorem.

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