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A Generalization of Auslander-Buchsbaum Theorem*
Auslander-Buchsbaum定理的推广

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Abstract The Auslander-Buchsbaum Theorem states that $pd_R M + Codim_R M = gl.dim R$ for each finitely generated nonzero module M over a Noetherian local ring R with finite global dimension. This theorem was generalized to nonzero finitely presented Noetherian modules M over a coherent local ring R with finitely generated maximal ideal J and finite weak global dimension([2]). Our aim is to extend the Auslander-Buchsbaum Theorem to any commutative coherent rings.

Key words coherent ring, finitely presented, global dimension, weak global dimension

摘要: Auslander-Buchsbaum定理指出,如果 R 是一个整体维数有限的 Noether局部环, M 是一个有限生成的非零 R-模,那么 $pd_R M + Codim_R M = gl.dim R$. 文献[2]证明上述公式对极大理想为有限生成的凝聚环上的有限表现的非零 Noether模依然成立.本文试图将 Auslander-Buchsbaum公式推广到任意的交换凝聚环上.

关键词: 凝聚环 有限表现 整体维数 弱整体维数
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1 Introduction

Throughout this paper it is assumed that all rings are commutative rings and all modules are unitary. Our aim in this paper is to extend the Auslander-Buchsbaum Theorem to any commutative coherent rings.

Let R be a ring and M a R-module. Recall that M is called finitely presented if there is a finitely generated R-module P and a finitely generated submodule N of P such that $P/N \cong M$. R is called a coherent ring if every finitely generated ideal of R is finitely presented.

In this paper, we use $Spec(R)$, $Max(R)$, $gl.dim R$, $w.gl.dim R$, $pd_R(M)$, $fd_R(M)$, $Codim_R(M)$ for the prime spectrum, the maximal spectrum, global dimension, weak global dimension of R, projective dimension, flat dimension, codimension of R-module M, respectively.

The well-known Auslander-Buchsbaum Theorem states that $pd_R M + Codim_R M = gl.dim R$ for each finitely generated nonzero module M over a Noetherian local ring R with finite global dimension[1]. In reference [2], it has been proved that the Auslander-Buchsbaum formula was true for a finitely presented Noetherian R-module M over a coherent local ring R with finitely generated maximal ideal and finite weak global dimension.

In Section 2, we extend this result to finitely presented (Noetherian) modules over any commutative coherent rings.

In Section 3, we give some examples and remarks on the main results in Section 2.

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2 Main results

Let R be a ring and M a R-module. We set $Z(M) = \{x \in R; xa = 0$ for some $a \neq 0$ in $M\}$, the set of zero divisors of M in R.

First we give several Lemmas.

Lemma 2.1 Let R be a ring and I a finitely generated ideal in R. Let M be a nonzero Noetherian R-module and N a proper submodule of M. If $I \subseteq Z(M/N)$ then there exists an element $a \in M$ but $a \notin N$ such that $Ia \subseteq N$.

In particular, if $I \subseteq Z(M)$ then there exists a nonzero element $a \in M$ such that $Ia = 0$.

Proof See reference [3, Theorem 1.3].

Lemma 2.2 Let R be a coherent ring, M a finitely presented R-module. For any natural numbers n, the following statements are equivament.

(a) $pd_R M \leqslant n$;

(b) $fd_R M \leqslant n$;

(c) $Tor^R_{n+i}(M, R/m) = 0$, for any $m \in$ Max (R) and any $i \geqslant 1$.

Proof See reference [4, Corollay 2.5.5 and Corollay 2.5.10].

By using of localization, from Lemma 2.2, it is easy to get the following consequence.

Corollary 2.3 Let R be a coherent ring, M a finitely presented R-module and n a non-negative integer. Then

(a) $pd_R M = fd_R M$;

(b) $fd_R M = n$ if and only if $fd_{R_m} M_m \leqslant n$ for all $m \in$ Max (R) and there exists at least one maximal ideal m such that $fd_{R_m} M_m = fd_R M = n$.

In the following discussion, the maximal ideals m which satisfying $fd_{R_m} M_m = fd_R M$ will play a very important role.

Lemma 2.4 Let R be a coherent ring and $m \in$ Max (R). Then

$$w.gl.dim R_m = fd_R(R/m) = fd_{R_m}(R_m/m_m).$$

Proof By reference [5, Theorem 5], $w.gl.dim R_m = fd_{R_m}(R_m/m_m)$ for R_m is a coherent local ring with unique maximal ideal m_m. So we need only to prove $fd_R(R/m) = fd_{R_m}(R_m/m_m)$. Since $m_{m'} = R_{m'}$ if $m \neq m' \in$ Max (R) and from reference [4, Theorem 1.3.14], we have

$fd_R(R/m) = \sup\{fd_{R_{m'}}(R/m)_{m'} \mid m' \in \text{Max}(R)\}$
$= \sup\{fd_{R_{m'}}(R_{m'}/m_{m'}) \mid m' \in \text{Max}(R)\} =$
$fd_{R_m}(R_m/m_m)$.

Definition 2.5 Let R be a ring, I an ideal of R and M a R-module. we define

I-$codim_R(M) = \sup\{n \mid T_1, \cdots, T_n$ is a regular M-sequence in $I\}$ and $codim_R(M) = \sup\{I$-$codim_R(M) \mid I$ is an ideal of $R\}$, which are called the codimension of M in I and codimension of M respectively. Since every ideal of R must be contained in a maximal ideal of R, we have $codim_R(M) = \sup\{m$-$codim_R(M) \mid m$ is a maximal ideal of $R\}$.

Theorem 2.6 Let R be a coherent ring and M a finitely presented R-module. If m is a maximal ideal of R satisfying $fd_{R_m} M_m = fd_R M = n < \infty$ and $T_1, \cdots, T_s$ is a regular M-sequence in m, then

$pd_R(M/(T_1, \cdots, T_s)M) = pd_R M + s$

and $pd_{R_m}(M_m/(T_1, \cdots, T_s)M_m) = pd_R(M/(T_1, \cdots, T_s)M)$.

Proof For a set M, we use id_M to denote the identity map of M. We prove it by induction on s.

If $s = 1$, then $T_1 \in m$ is a regular M-sequence and thus there exists a short exact sequence

$$0 \longrightarrow M \xrightarrow{f} M \to M/T_1 M \to 0, \quad (1)$$

where $f(x) = T_1 x, \forall x \in M$. So we have a long exact sequence

$$\cdots \to Tor^R_{j+1}(M, R/m) \xrightarrow{H_{j+1}} Tor^R_{j+1}(M, R/m) \to Tor^R_{j+1}(M/T_1 M, R/m) \xrightarrow{\Delta} Tor^R_j(M, R/m) \xrightarrow{H_j} Tor^R_j(M, R/m) \to \cdots$$

$\forall j \geqslant 0$, $H_j = Tor^R_j(f, id_{R/m}) = Tor^R_j(T_1 id_M, id_{R/m}) = Tor^R_j(id_M, T_1 id_{R/m}) = Tor^R_j(id_M, 0) = 0$.

Thus, $\forall j \geqslant 0$, we have a short exact sequence

$$0 \to Tor^R_{j+1}(M, R/m) \to Tor^R_{j+1}(M/T_1 M, R/m) \to Tor^R_j(M, R/m) \to 0. \quad (2)$$

Since $fd_R M = fd_{R_m} M_m = n < \infty$, $Tor^R_{n+1}(M, R/m) = 0$ and $Tor^R_n(M_m, R_m/m_m) \neq 0$. Hence $Tor^R_n(M, R/m) \neq 0$. Thus from Sequence (2), $Tor^R_{n+1}(M/T_1 M, R/m) \neq 0$ and therefore, by Lemma 2.2, $pd_R(M/T_1 M) \geqslant n + 1 = pd_R M + 1$. By Sequence (1) and reference [6, Lemma 9.26], $pd_R(M/T_1 M) \leqslant pd_R M + 1 = n + 1$. Hence $pd_R(M/T_1 M) = pd_R M + 1$. From the above discussion, we also have $pd_R(M/T_1 M) = pd_{R_m}(M/T_1 M)_m = n + 1$.

Now suppose $s > 1$, $T_1, \cdots, T_s$ is a M-sequence in m. Set $M^* = M/T_1 M$. By the above proof, $pd_R M^* = pd_{R_m}(M^*)_m = pd_R M + 1$ and it is clear that $T_2, \cdots, T_s$ is a regular M^*-sequence in m. Hence, by the inductive hypothesis, we have $pd_R(M^*/(T_2, \cdots, T_s)M^*) = pd_R M + s - 1 = pd_R M + s$, $M^*/(T_2, \cdots, T_s)M^* = (M/T_1 M)/((T_2, \cdots, T_s)M/T_1 M) \cong M/(T_1, \cdots, T_s)M$ and $pd_R(M^*/(T_2, \cdots, T_s)M^*) = pd_{R_m}(M^*_m/(T_2, \cdots, T_s)M^*_m)$.

Thus $pd_R(M/(x_1,\cdots,x_s)M) = pd_RM + s$ and $pd_R(M/(x_1,\cdots,x_s)M) = pd_{R_m}(M_m^* /(x_2,\cdots,x_s)M_m^*) = pd_{R_m}(M_m/(x_1,\cdots,x_s)M_m)$.

Lemma 2.7 Let R be a coherent ring and M a finitely presented noetherian R-module and m a maximal ideal of R satisfying $pd_{R_m}M_m = pd_RM$. If $w.gl.dim R_m < \infty$ then $pd_RM = w.gl.dim R_m$ if and only if there exists a nonzero submodule N of M such that $mN = 0$.

Proof If there exists a non-zero submodule N of M such that $mN = 0$, we may assume $N = (x)$. The homomorphism $R \to N$, in which $r \in R$ is mapped into rx, has kernel m, hence $N \cong R/m$, and then we have an exact sequence $0 \to R/m \to M \to M/N \to 0$. By localization, the sequence

$$0 \to R_m/m_m \to M_m \to M_m/N_m \to 0 \qquad (3)$$

is also exact. Since $w.gl.dim R_m = n < \infty$, by Lemma 2.4, $w.gl.dim R_m = fd_{R_m}(R_m/m_m)$ and by reference [4, Theorem 1.3.9], $w.gl.dim R_m = \sup\{fd_{R_m}(R_m/I) | I$ a finitely generated ideal of $R_m\}$ and by referencee [7, Theorem 1.2.5], $fd_{R_m}(R_m/I) = \sup\{i | Tor_i^R(R_m/I, R_m/m_m) \neq 0\}$, thus there exists a finitely generated ideal I of R_m such that $w.gl.dim R_m = fd_{R_m}(R_m/I) = n$ and then $Tor_n^{R_m}(R_m/I, R_m/m_m) \neq 0$ and $Tor_{n+1}^{R_m}(K, R_m/m_m) = 0$ for any R_m-module K. Write the long exact sequence of $Tor^{R_m}(R_m/I, -)$ resulting from the short exact sequence (3) to obtain exact sequence $0 \to Tor_n^{R_m}(R_m/I, R_m/m_m) \to Tor_n^{R_m}(R_m/I, M_m)$ and then $Tor_n^{R_m}(R_m/I, M_m) \neq 0$, which shows that $fd_{R_m}M_m \geqslant n$. But $w.gl.dim R_m = n$. Hence $fd_{R_m}M_m = w.gl.dim R_m = n$ and then $pd_RM = w.gl.dim R_m$.

Now suppose $pd_RM = w.gl.dim R_m$, we need to prove there exists a nonzero submodule N of M such that $mN = 0$. Otherwise, by Lemma 2.1, there exists $x \in m$ such that x is not a zero-divisor on M and then by Theorem 2.6, $pd_{R_m}(M_m/xM_m) = pd_RM/xM = pd_RM + 1 = w.gl.dim R_m + 1 > w.gl.dim R_m$. This contradiction completes our proof.

Now, we can prove our main result, which is a generalization of Auslander-Buchsbaum Theorem over any commutative rings.

Theorem 2.8 Let R be a coherent ring, M a nonzero R-module and m a maximal ideal of R satisfying $pd_{R_m}M_m = pd_RM$.

(1) If M is finitely presented, then $pd_RM + m\text{-}codim_RM \leqslant w.gl.dim R_m$;

(2) If M is a finitely presented Noetherian R-module and $w.gl.dim R_m < \infty$, then $pd_RM + m\text{-}codim_RM = w.gl.dim R_m$.

Proof (1) Let $x_1,\cdots,x_s$ be any M-sequences in m. By Theorem 2.6,

$pd_RM + s = pd_R(M/(x_1,\cdots,x_s)M) = pd_RM + s = pd_{R_m}(M_m/(x_1,\cdots,x_s)M_m) \leqslant w.gl.dim R_m$, which implies that $pd_RM + m - codim_RM \leqslant w.gl.dim R_m$.

(2) By (1), $m\text{-}codim_RM \leqslant w.gl.dim R_m < \infty$. Suppose $m\text{-}codim_RM = s$. Then there exists a regular M-sequence $x_1,\cdots,x_s$ in m. By Theorem 2.6, we have $pd_RM + m\text{-}codim_RM = pd_RM + s = pd_R(M/(x_1,\cdots,x_s)M)$. Now we need only to show that $pd_R(M/(x_1,\cdots,x_s)M) = w.gl.dim R_m$. Thus, by Lemma 2.7, we need only to prove m annihilates some nonzero submodule of $M/(x_1,\cdots,x_s)M$. Otherwise, since M is a finitely presented Noetherian R-module, so is $M/(x_1,\cdots,x_s)M$. By Lemma 2.1, there exists $x_{s+1} \in m$ such that x_{s+1} is not a zero divisor on $M/(x_1,\cdots,x_s)M$. This means that $x_1,\cdots,x_s,x_{s+1}$ is a regular M-sequence in m, and therefore $m\text{-}codim_RM \geqslant s+1$, which contradicts the above hypothesis.

Hence $pd_RM + m\text{-}codim_RM = w.gl.dim R_m$.

Corollary 2.9 Let R be a coherent local ring and M a nonzero R-module. Then

(1) If M is a non-zero finitely presented R-module, then $pd_RM + codim_RM \leqslant w.gl.dim R$.

(2) If M is a non-zero finitely presented Noetherian R-module and $w.gl.dim R < \infty$, then $pd_RM + codim_RM = w.gl.dim R$.

Corollary 2.10 Let R be a Noetherian ring and M a non-zero finitely generated R-module and $m \in \mathrm{Max}(R)$ satisfying $pd_RM = pd_{R_m}M_m$ and $gl.dim R_m < \infty$. Then $pd_RM + m\text{-}codim_RM = gl.dim R_m$.

Corollary 2.11 Let R be a coherent ring and M a non-zero finitely presented Noetherian R-module. If $m \in \mathrm{Max}(R)$ satisfies $pd_RM = pd_{R_m}M_m$ and $w.gl.dim R_m < \infty$, then $m\text{-}codim_RM = codim_{R_m}M_m$.

Proof By Theorem 2.8(2), we have $pd_RM + m\text{-}codim_RM = w.gl.dim R_m$. From Corollary 2.9, we have $pd_{R_m}M_m + codim_{R_m}M_m = w.gl.dim R_m$. From the assumption $pd_RM = pd_{R_m}M_m$ and the above two equations, we get $m\text{-}codim_RM = codim_{R_m}M_m$.

Corollary 2.12 Let (R,m) be a coherent local ring with maximal ideal m and P a finitely generated prime ideal properly contained in m.

(1) If $w.gl.dim R = 2$, then P must be projective, therefore P must be a principal ideal;

(2) If $w.gl.dim R = 3$, then $pd_RP \leqslant 1$.

Proof (1) Let $M = R/P$. By the hypothesis, there exists $T \in m - P$. Obviously, T is not a zero divisor on M for P is a prime ideal. Thus $codim_R R/P \geqslant 1$ and by Corollary 2.9(1), $pd_R R/P \leqslant 1$. The exactness of the short sequence $0 \to P \to R \to R/P \to 0$ shows that $pd_R P = 0$ and therefore P is a projective R-module and therefore P is free for R is a local ring. The fact that R is a domain implies that P is a principal ideal.

(2) The proof of (2) is similar to that of (1), we omit it.

3 Examples and remarks

Example 3.1 Let (R, m) be an umbrella ring with unique maximal ideal m (see reference [8] for the definition). By reference [9, Example 3], we know that R is a coherent local domain and $w.gl.dim\ R = gl.dim R = 2$ and m can be generated by a two-element R-sequence $\{T_1, T_2\}$ and there is a maximal non-finitely generated prime ideal P of R. So R is a non-Noetherian coherent local ring.

(1) Let $M = R/m$. Then M is a finitely presented Noetherian R-module. It is easy to verify that $codim_R M = 0, pd_R M = fd_R M = 2$, and hence $pd_R M + codim_R M = w.gl.dim\ R$.

(2) Let $M = R/(T)$, where (T) is a principal prime ideal of R. Then M is also a finitely presented Noetherian R-module. We can verify that $codim_R M = 1, pd_R M = fd_R M = 1$ and the equality $pd_R M + codim_R M = w.gl.dim\ R$ holds.

(3) Let $M = m = (T_1, T_2)$. Then M is a finitely presented non-Noetherian R-module and it is easy to verify that $codim_R M = 1$ and $pd_R M = fd_R M = 1$. Hence $pd_R M + codim_R M = w.gl.dim\ R$.

(4) Let $M = (T)$ be a principal ideal of R. Then $M \simeq R$ is a finitely presented non-Noetherian R-module and obviously $codim_R M = 2, pd_R M = 0$ and therefore $pd_R M + codim_R M = w.gl.dim\ R$ also holds.

(5) Let $M = R/P$, where P is the maximal non-finitely generated prime ideal of R. Then M is a Noetherian R-module but not a finitely presented R-module by reference [6, Corollary 3.63]. It is easy to verify that $codim_R M = 2$. By reference [4, Theorem 6.3.3], P is a flat R-module. If R/P is flat then it is free by reference [10, Theorem 7.10], which implies that P is a principal ideal of R, a contradiction. Hence $fd_R R/P \geqslant 1$ and therefore $pd_R M + codim_R M \geqslant 1 + 2 = 3 > w.gl.dim R$.

Example 3.2 Let $R = \{f(x) \in Q[x] \mid f(0) \in Z\}$, where Q is the field of rational numbers and Z is the ring of integral numbers. By reference [11, Theorem 1 and Theorem 2], we have $Max(R) = \{pR \mid p$ is a prime number in $Z\} \cup \{p(x)R \mid p(x)$ is an irreducible polynomial $\in Q[x]$ and $p(0) = 1$ or $-1\}$ and $Spec R = Max(R) \cup \{P_\infty, 0\}$, where $P_\infty = \{f(x) \in R \mid f(0) = 0\}$ is the unique non-finitely generated prime ideal of R and $P_\infty \subset pR$ for any prime numbers $p \in Z$ and $w.gl.dim\ R = 1 = w.gl.dim\ R_m, \forall m \in Max(R)$ and R is a Bezout ring (therefore a coherent ring). Clearly, R is not a semilocal ring.

(1) $\forall T \in R, T \neq 0$, let $M = R/(T)$. Obviously, M is a finitely presented R-module and $pd_R M = 1$. By using of Theorem 2.8(2), we can verify that if $m \in Max(R)$ satisfies $pd_{R_m} M_m = pd_R M$ then $m\text{-}codim_R M = 0$. Hence the equality $pd_R M + m\text{-}codim_R M = w.gl.dim\ R_m$ holds. (Notice that $M = R/(T)$ may be Noetherian or non-Noetherian. For example, $R/(8) \simeq Z/(8)$ is a Noetherian, but $R/(x)$ is not a Noetherian R-module).

(2) Let $M = R/P_\infty$. Then M is a Noetherian R-module. By reference [7, Corollary 3.63], M is not finitely presented. We can also verify that for any prime numbers $p \in Z, fd_{R_{(p)}} M_{(p)} = fd_R M = 1$ and $(p)\text{-}codim_R M = 1$. Thus $fd_R M + (p)\text{-}codim_R M = 2 > w.gl.dim R_{(p)}$.

By observing the above examples, we have the following remarks:

Remark 3.1 If M is a Noetherian R-module but not finitely presented, then Theorem 2.8(2) and Corollary 2.9(2) will not hold. See Example 3.1(5) and Example 3.2(2).

Remark 3.2 So far, we can not find a coherent ring R and a nonzero finitely presented non-Noetherian R-module M and a maximal ideal m of R satisfying $pd_R M = pd_{R_m} M_m$ and $w.gl.dim R_m < \infty$, but $fd_R M + m\text{-}codim_R M < w.gl.dim R_m$. See Example 3.1(3), (4) and Example 3.2(1). So we have a question: without the assumption "Noetherian" does the Theorem 2.8(2) hold?

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$R^H R = T \operatorname{diag}(\lambda_1, \cdots, \lambda_t, 0, \cdots, 0) T^H$,

where $T = (t_{ij})_{n\times n}$ is a unite matrix, and $\lambda_1 > 0$.

Since $C = (c_{ij})_{n\times n} = T^H R^H RDR^H RDT$, $G = (g_{ij})_{n\times n} = T^H R^H ERT$ are non-negative definite matrices, then $c_{11} \leqslant 0$, $g_{11} \geqslant 0$.

Let $F = \operatorname{diag}(\lambda_1, \cdots, \lambda_t, 0, \cdots, 0) T^H DT + T^H DT \operatorname{diag}(\lambda_1, \cdots, \lambda_t, 0, \cdots, 0)$,

we obtain $f_{11} = 2\lambda_1 (\sum_{i=1}^{n} d_i t_{1i} t_{1i}^H)$.

From $\sum_{i=1}^{n} t_{1i} t_{1i}^H = 1, d_i > 0$, we have

$f_{11} + c_{11} + g_{11} > 0$,

which is a contradiction to Formula (2.6).

So $R = 0$, and $A^H \leqslant B$.

Corollary 2.8 Let A, B be non-negative definite matrices, if $A \leqslant B$, $A^3 \leqslant B^3$, then $AB = BA$.

Proof If $A \leqslant B$, $A^3 \leqslant B^3$, then $A^H \leqslant B$, so $AB = BA$.

3 Conclude

The relation between the minus partial ordering of two matrices A and B relates to the B- H partial ordering of theirs exponent A^k and B^k $(k = 2, 3)$ are given, But our method seems unavailable for the general case, and we pose an open question.

Question As a consequence of above corollary, we conjecture

$A \leqslant B, A^k \leqslant B^k (k \geqslant 4) \Rightarrow AB = BA$.

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