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A Classification of Quadratic Harmonic Morphisms Between Semi-Euclidean Spaces $R^3_r \to R^2_s$*

二次调和同态 φ: $R^3_r \to R^2_s$ 的分类

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Abstract In this paper, we study quadratic harmonic morphisms between semi-Euclidean spaces. We give a structure equation of such morphisms using their coefficient matrices analysis and special coordinates generalizing the results of Ou-Wood on quadratic harmonic morphisms between Euclidean spaces. As an application, we obtain a classification of quadratic harmonic morphisms $R^3_r \to R^2_s$.

Key words semi-Euclidean spaces R^m_r, harmonic morphisms, quadratic harmonic morphisms.

摘要:在给出半定欧氏空间之间二次调和同态的结构方程之后,通过对结构方程的系数矩阵的分析及特殊坐标系的运用,推广 Ou-Wood 关于欧氏空间之间二次调和同态的结果,获得二次调和同态 $R^3_r \to R^2_s$ 的分类.

关键词:半欧氏空间 R^m_r 调和同态 二次调和同态

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Unless otherwise stated, all manifolds and maps involved in this paper are assumed to be smooth.

Definition 0.1 A map $\varphi: (M,g) \to (N,h)$ between two Riemannian manifolds is called a harmonic map if the divergence of its differential vanishes. Such maps are the critical points of the energy function

$$E_2(\varphi,\Omega) = \frac{1}{2}\int_\Omega |d\varphi|^2 dx$$

over compact domain Ω in M. For further detailed account on harmonic maps, we refer to References [1~3].

The function corresponding to the Euler-Lagrange equation is the tension fields, a system of semi-linear second order elliptic partial differential forms:

$$\tau(\varphi) = trace \nabla d\varphi.$$

In local coordinates, harmonic map equation takes the form

$$\tau^\gamma = \Delta_M \varphi^\gamma + \sum_{\alpha,\beta=1}^{n} g_M(\nabla\varphi^\alpha, \nabla\varphi^\beta)({}^N\Gamma^\gamma_{\alpha\beta}\circ\varphi) = g^{ij}\left(\frac{\partial^2\varphi^\gamma}{\partial x^i\partial x^j} - {}^M\Gamma^k_{ij}\frac{\partial\varphi^\gamma}{\partial x^k} + {}^N\Gamma^\gamma_{\alpha\beta}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}\right),$$

where ${}^M\Gamma^k_{ij}$ and ${}^N\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols of the Levi-Civita connections on (M,g) and (N,h). φ is called harmonic map if $\tau(\varphi) = 0$.

Harmonic morphisms are defined as mappings between Riemannian manifolds which pull back (local) harmonic functions to (local) harmonic functions. More precise is as follow:

Definition 0.2 Let $\varphi: (M,g) \to (N,h)$ be a mapping between Riemannian manifolds. Then φ is called a harmonic morphism if for any harmonic function $f: U \subset N \to R$ with $\varphi^{-1}(U)$ non-empty, its pull-back by $f\circ\varphi: \varphi^{-1}(U) \subset M \to R$ is harmonic as well.

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Definition 0.3 A C^1 - map $\varphi:(M^m,g)\to(N^n,h)$ is called horizontally weak conformal map if, at each point $x\in M$, either $d\varphi=0$ or the linear map

$$d\varphi_{(ker d\varphi)^{\perp}}:(ker d\varphi)^{\perp}\to T_{\varphi(x)}N$$

is conformal and surjective. Let $V_x=ker\,d\varphi_x$ and $H_x=(ker d\varphi_x)^{\perp}$, then horizontally weak conformality can be written as

$$h(d\varphi(X),d\varphi(Y))=\lambda^2(x)g(X,Y),\forall X,Y\in H_x.$$

We call λ the dilation of horizontally weak conformal map φ. It is easily seen that in local coordinates $(x^i)_{i=1,2,\cdots,m}$ and $(y^{\alpha})_{\alpha=1,2,\cdots,n}$ around x and $\varphi(x)$, the horizontal weak conformality reads:

$$g^{ij}(x)\frac{\partial\varphi^{\alpha}}{\partial x^i}(x)\frac{\partial\varphi^{\beta}}{\partial x^j}(x)=\lambda^2(x)h^{\alpha\beta},\forall X,Y\in H_x.$$

It is well known that a mapping between Riemannian manifolds is a harmonic morphism if it is a horizontally weak conformal harmonic map[4,5].

In 1996, Fuglede[4] extended the above characterization to harmonic morphisms between semi-Riemannian manifolds. For an early study of harmonic morphism between semi-Riemannian manifolds see Reference[6].

Recall that a semi-Riemannian manifold[7], the Laplace-Beltrami operator Δ_M is not elliptic in general, in local coordinates $(x^i)_{i=1,2,\cdots,m}$ around $x\in M^m$, given by

$$\Delta_M=\frac{1}{\sqrt{|g_M|}}\sum_{i=1}^{m}\frac{\partial}{\partial x^i}\left(\sqrt{|g_M|}\sum_{j=1}^{m}g_M^{ij}\frac{\partial}{\partial x^j}\right),$$

where $g_M=det(g_{ij}^M)$, g_{ij}^M and g_M^{ij} being the covariant and the contra-variant components of the metric tensor g_M. A harmonic morphism between semi-Riemannian manifolds M,N is defined as a smooth map $M\to N$ which pulls back local harmonic functions on N into local harmonic functions on M. As in the Riemannian case, a harmonic morphism is the same as a smooth map which is harmonic and horizontally weakly conformal.

In this paper, we focus on the study of harmonic morphisms between semi-Euclidean spaces. We use R_r^m to denote the semi-Euclidean space which is R^m as manifolds and it is provided with the indefinite metric g_r^m given by

$$(g_{ij})\equiv\begin{pmatrix}-I_r & O\\ O & I_{m-r}\end{pmatrix}$$

where (g_{ij}) denotes the component matrix of tensor g_r^m and I_r denotes the standard $r\times r$ identity matrix.

Definition 0.4 A map $\varphi:R_r^m\to R_s^n$ is called a quadratic map if all component functions of φ are homogeneous polynomials of degree 2 in $x_1,x_2,\cdots,x_m$. In this case, from the theory of quadratic functions and bi-linear forms we know that a quadratic map $\varphi:R_r^m\to R_s^n$ can be always written as

$$\varphi=(X^tA_1X,X^tA_2X,\cdots,X^tA_nX)$$

where X denotes the column vector in R_r^m, X^t the transpose of X and the symmetric matrices A_i $(i=1,2,\cdots,n)$ are called the component matrices.

Theorem 0.1[8] A quadratic map $R^m\to R^n$ $(m\geqslant n)$ (i.e. $R_0^m\to R_0^n$) with

$$\varphi(X)=(X^tA_1X,X^tA_2X,\cdots,X^tA_nX)$$

is a harmonic morphism if and only if

(1) $tr\ A_i=0,(i=1,2,\cdots,n)$;

(2) $A_iA_j+A_jA_i=O,(i,j=1,2,\cdots,n,i\neq j)$;

(3) $A_i^2=A_j^2,(i,j=1,2,\cdots,n)$.

The following characterization of harmonic morphisms was obtained in Reference [9].

Theorem 0.2 For a map $\varphi:U\subset R_r^m\to R_s^n$ between semi-Euclidean spaces with

$$\varphi(x)=(\varphi^1(x),\varphi^2(x),\cdots,\varphi^n(x)),$$

the harmonicity and horizontally weak conformality are equivalent to the following conditions respectively

$$-\sum_{i=1}^{r}\frac{\partial^2\varphi^{\alpha}}{\partial x_i^2}+\sum_{i=r+1}^{m}\frac{\partial^2\varphi^{\alpha}}{\partial x_i^2}=0, \qquad (0.1)$$

$$-\sum_{i=1}^{r}\frac{\partial\varphi^{\alpha}}{\partial x_i}\frac{\partial\varphi^{\beta}}{\partial x_i}+\sum_{i=r+1}^{m}\frac{\partial\varphi^{\alpha}}{\partial x_i}\frac{\partial\varphi^{\beta}}{\partial x_i}=\lambda^2\varepsilon_{\alpha}\delta_{\alpha\beta}, \qquad (0.2)$$

where $\alpha,\beta=1,2,\cdots,n$; $(x_1,x_2,\cdots,x_m)$ are standard coordinates of R_r^m; $\lambda:R_r^m\to R$ is the dilation of φ, and

$$\varepsilon_{\alpha}=\begin{cases}-1, & \alpha=1,2,\cdots,s\\ 1, & \alpha=s+1,\cdots,n.\end{cases}$$

1 Equations for quadratic Harmonic

Proposition 1.1 Let $\varphi:R_1^3\to R_0^2$ be a quadratic harmonic morphism with

$$\varphi(X)=(X^tAX,X^tBX),$$

then

(i) $trA=0,trB=0$,

where the trace is taken with respect to the metric g_r^m, for instance, $A=(a_{ij})$, then trace of A can be written as

$$trA=-\sum_{i=1}^{r}a_{ii}+\sum_{i=r+1}^{m}a_{ii}.$$

(ii) $AI_1^3B + BI_1^3A = O$,

where $I_1^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

(iii) $AI_1^3A = BI_1^3B$.

Proof Using the expression of a harmonic morphism between semi-Euclidean spaces $\varphi: R_1^3 \to R_0^2$ given by

$$\varphi(x_1,x_2,x_3) = (a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3, b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + 2b_{12}x_1x_2 + 2b_{13}x_1x_3 + 2b_{23}x_2x_3),$$

$$\varphi^1(x_1,x_2,x_3) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3, \quad (1.1)$$

$$\varphi^2(x_1,x_2,x_3) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + 2b_{12}x_1x_2 + 2b_{13}x_1x_3 + 2b_{23}x_2x_3, \quad (1.2)$$

we can find out the equations of coefficients a_{ij} and b_{ij} $(i,j = 1, 2, 3)$.

From Equation (0.1), we get

$$-\frac{\partial^2\varphi^1}{\partial x_1^2} + \frac{\partial^2\varphi^1}{\partial x_2^2} + \frac{\partial^2\varphi^1}{\partial x_3^2} = 0,$$

$$-\frac{\partial^2\varphi^2}{\partial x_1^2} + \frac{\partial^2\varphi^2}{\partial x_2^2} + \frac{\partial^2\varphi^2}{\partial x_3^2} = 0$$

and

$$-a_{11} + a_{22} + a_{33} = 0, \; -b_{11} + b_{22} + b_{33} = 0,$$

which implys that

$$trA = 0, trB = 0.$$

According to Equation (0.2), we have

$$-\frac{\partial\varphi^\alpha}{\partial x_1}\frac{\partial\varphi^\alpha}{\partial x_1} + \frac{\partial\varphi^\alpha}{\partial x_2}\frac{\partial\varphi^\alpha}{\partial x_2} + \frac{\partial\varphi^\alpha}{\partial x_3}\frac{\partial\varphi^\alpha}{\partial x_3} = \lambda^2, \alpha = 1, 2. \quad (1.3)$$

$$-\frac{\partial\varphi^\alpha}{\partial x_1}\frac{\partial\varphi^\beta}{\partial x_1} + \frac{\partial\varphi^\alpha}{\partial x_2}\frac{\partial\varphi^\beta}{\partial x_2} + \frac{\partial\varphi^\alpha}{\partial x_3}\frac{\partial\varphi^\beta}{\partial x_3} = 0, \alpha,\beta = 1, 2, \alpha \neq \beta. \quad (1.4)$$

Substituting (1.1) and (1.2) into (1.3), we have

$$(-a_{11}^2 + a_{12}^2 + a_{13}^2)x_1^2 + (-a_{12}^2 + a_{22}^2 + a_{23}^2)x_2^2 + (-a_{13}^2 + a_{23}^2 + a_{33}^2)x_3^2 + 2(-a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{23})x_1x_2 + 2(-a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33})x_1x_3 + 2(-a_{12}a_{13} + a_{22}a_{23} + a_{23}a_{33})x_2x_3 = \frac{\lambda^2}{4},$$

expressed by matrix, we get

$$(x_1,x_2,x_3)\begin{pmatrix} -a_{11} & a_{12} & a_{13} \\ -a_{12} & a_{22} & a_{23} \\ -a_{13} & a_{23} & a_{33} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{\lambda^2}{4},$$

in the same way, we have

$$(x_1,x_2,x_3)\begin{pmatrix} -b_{11} & b_{12} & b_{13} \\ -b_{12} & b_{22} & b_{23} \\ -b_{13} & b_{23} & b_{33} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{\lambda^2}{4},$$

therefore

$$AI_1^3A = BI_1^3B.$$

Expanding (1.4), we have

$$(-a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13})x_1^2 + (-a_{12}b_{12} + a_{22}b_{22} + a_{23}b_{23})x_2^2 + (-a_{13}b_{13} + a_{23}b_{23} + a_{33}b_{33})x_3^2 + 2[(-a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{23}) + (-b_{11}a_{12} + b_{12}a_{22} + b_{13}a_{23})]x_1x_2 + 2[(-a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}) + (-b_{11}a_{13} + b_{12}a_{23} + b_{13}a_{33})]x_1x_3 + 2[(-a_{12}b_{13} + a_{22}b_{23} + a_{23}b_{33}) + (-b_{12}a_{13} + b_{22}a_{23} + b_{23}a_{33})]x_2x_3 = 0,$$

which implies that

$$(x_1,x_2,x_3)\begin{pmatrix} -a_{11} & a_{12} & a_{13} \\ -a_{12} & a_{22} & a_{23} \\ -a_{13} & a_{23} & a_{33} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (x_1,x_2,x_3)AI_1^3B\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

Noting that AI_1^3B is not symmetric in general, we conclude that

$$AI_1^3B + BI_1^3A = O.$$

Thus we end the proof of the lemma.

In a similar way, we can obtain more general results in the following

Proposition 1.2 Let $\varphi: R_r^m \to R_s^n (m \geqslant n)$ be a quadratic map given by

$$\varphi(X) = (X^tA_1X, X^tA_2X, \cdots, X^tA_nX),$$

then φ is a harmonic morphism if and only if

(1) $trA_i = 0, (i = 1, 2, \cdots, n)$,

where trace of A_i is taken with respect to the metric g_r^m,

(2) $A_iI_r^mA_j + A_jI_r^mA_i = O, (i,j = 1, 2, \cdots, n; i \neq j)$,

(3) $\varepsilon_iA_iI_r^mA_i = \varepsilon_jA_jI_r^mA_j, (i,j = 1, 2, \cdots, n)$.

for $i,j = 1, 2, \cdots, n$, where

$$I_r^m = \begin{pmatrix} -I_r & O \\ O & I_{m-r} \end{pmatrix}.$$

Example 1.1[9] Let $\varphi: R_2^4 \to R_2^3$ by

$$\varphi(x^1,x^2,x^3,x^4) = (2x^1x^3 - 2x^2x^4, 2x^1x^4 + 2x^2x^3, (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2),$$

then φ is a harmonic morphism defined by homogeneous polynomial of degree 2 with dilation $4|x|^2$.

In this example, it is well known that

$$A_1=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix},$$
$$A_2=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},$$
$$A_3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

By a routine computation, we can easily get

$trA_1=0, trA_2=0, trA_3=0.$

$A_1I_2^4A_2+A_2I_2^4A_1=O,$

$A_1I_2^4A_3+A_3I_2^4A_1=O,$

$A_2I_2^4A_3+A_3I_2^4A_2=O,$

$\varepsilon_1A_1I_2^4A_1=\varepsilon_2A_2I_2^4A_2,$

$\varepsilon_1A_1I_2^4A_1=\varepsilon_3A_3I_2^4A_3,$

$\varepsilon_1A_1I_2^4A_2=\varepsilon_3A_3I_2^4A_3$, where $\varepsilon_1=-1, \varepsilon_2=-1, \varepsilon_3=1$.

2 Some classifications of quadratic harmonic morphism $\varphi: R_r^3\rightarrow R_s^2$

Proposition 2.1 Let $\varphi: R_1^3\rightarrow R_0^2$ be a quadratic harmonic morphism, then, up to an isometry of R^3, φ is the composition of an orthogonal projection

$\pi: R_1^3\rightarrow R_0^2, \pi(x_1,x_2,x_3)=(0,x_2,x_3)$,

followed by a quadratic harmonic morphism $\psi: R_0^2\rightarrow R_0^2$.

Proof Let $\varphi(X)=(X^tAX, X^tBX)$. After a suitable choice of orthogonal coordinates in R^3 is done, A takes the diagonal form

$$A=\begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix},$$

where $\lambda_1^2+\lambda_2^2+\lambda_3^2\neq 0$.

Using Equations (i), (ii) and (iii) of Proposition 1.1, we have

$$-\lambda_1+\lambda_2+\lambda_3=0,\ -b_{11}+b_{22}+b_{33}=0, \tag{2.1}$$

$$\begin{pmatrix}-2\lambda_1b_{11}&(-\lambda_1+\lambda_2)b_{12}&(-\lambda_1+\lambda_3)b_{13}\\(-\lambda_1+\lambda_2)b_{12}&2\lambda_2b_{22}&(\lambda_2+\lambda_3)b_{23}\\(-\lambda_1+\lambda_3)b_{13}&(\lambda_2+\lambda_3)b_{23}&2\lambda_3b_{33}\end{pmatrix}=O, \tag{2.2}$$

and

$$\begin{pmatrix}-\lambda_1^2&0&0\\0&\lambda_2^2&0\\0&0&\lambda_3^2\end{pmatrix}=\begin{pmatrix}-b_{11}&b_{12}&b_{13}\\-b_{12}&b_{22}&b_{23}\\-b_{13}&b_{23}&b_{33}\end{pmatrix}\begin{pmatrix}b_{11}&b_{12}&b_{13}\\b_{12}&b_{22}&b_{23}\\b_{13}&b_{23}&b_{33}\end{pmatrix}. \tag{2.3}$$

It is easy to check that none of the following cases is held

(i) $\lambda_1\lambda_2\lambda_3\neq 0$,

(ii) $\lambda_i^2+\lambda_j^2=0, i\neq j, i,j=1,2,3$,

(iii) $\lambda_2=0$ but $\lambda_1\lambda_3\neq 0$, or $\lambda_3=0$ but $\lambda_1\lambda_2\neq 0$.

Consider the case $\lambda_1=0, \lambda_2\neq 0, \lambda_3\neq 0$. From (2.1) we get

$\lambda_2+\lambda_3=0,\ -\lambda_1+\lambda_2\neq 0,\ -\lambda_1+\lambda_3\neq 0.$

Let $\lambda_2=\lambda, \lambda_3=-\lambda$, from (2.2), it follows that

$b_{12}=b_{13}=b_{22}=b_{33}=0.$

Combining (2.3) we see that

$b_{11}=0.$

Thus,

$$B=\begin{pmatrix}0&0&0\\0&0&b_{23}\\0&b_{23}&0\end{pmatrix}.$$

Again from (2.3) it follows that

$b_{23}=\pm\lambda.$

Therefore, we obtain

$$A=\begin{pmatrix}0&0&0\\0&\lambda&0\\0&0&-\lambda\end{pmatrix}, B=\begin{pmatrix}0&0&0\\0&0&\pm\lambda\\0&\pm\lambda&0\end{pmatrix}.$$

This means that φ is the composition $\psi\circ\pi$ of π and ψ. Thus we obtain Proposition 2.1.

In a similar way, for such as $\varphi: R_0^3\rightarrow R_0^2, \varphi: R_2^3\rightarrow R_0^2, \varphi: R_3^3\rightarrow R_0^2, \varphi: R_2^3\rightarrow R_2^2$, we have the same conclusion.

Proposition 2.2 Any quadratic harmonic morphisms between semi-Euclidean spaces $\varphi: R_1^3\rightarrow R_1^2$ is, up to an isometry of R^3, the composition of an orthogonal projection $\pi: R_1^3\rightarrow R_1^2$ followed by a quadratic harmonic morphism $\psi: R_1^2\rightarrow R_1^2$.

Proof Let $\varphi(X)=(X^tAX, X^tBX)$. According to Proposition 1.2, we have

1) $trA=0, trB=0$,

2) $AI_1^3B+BI_1^3A=O$,

3) $-AI_1^3A=BI_1^3B$.

We can choose a suitable coordinates, such that A

can be written

$$A=\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix},$$

A direct checking shows that the following cases are impossible

(i) $\lambda_1\lambda_2\lambda_3\neq 0$,

(ii) $\lambda_1=0$ but $\lambda_2\lambda_3\neq 0$,

(iii) $\lambda_i^2+\lambda_j^2=0, i\neq j$.

Then we are left with the case that $\lambda_3=0$, but $\lambda_1\lambda_2\neq 0$ (or $\lambda_2=0$ but $\lambda_1\lambda_3\neq 0$).

By 1) we have

$\lambda_1=\lambda_2=\lambda\neq 0, -\lambda_1+\lambda_3\neq 0, \lambda_2+\lambda_3\neq 0$.

By 2) we get

$b_{11}=b_{13}=b_{22}=b_{23}=0$.

Using 1) again we deduce that

$b_{33}=0$.

From 3) it follows that

$b_{12}=\pm\lambda$.

Thus, we have

$$A=\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & 0\end{pmatrix}, B=\begin{pmatrix}0 & \pm\lambda & 0\\ \pm\lambda & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

It follows that $\varphi\circ\pi$ with π being orthogonal projection defined by $\pi(x_1,x_2,x_3)=(x_1,x_2,0)$ and φ being an quadratic harmonic morphism.

Similarly, for $\varphi: R_2^3\to R_1^2$, as long as we take the orthogonal projection as $\pi: R_2^3\to R_1^2$, defined by $\pi(x_1,x_2,x_3)=(0,x_2,x_3)$, we can get the same result as Proposition 2.2.

The above results can be summarized as follows

Theorem 2.1 Let $\varphi: R_r^3\to R_s^2$ be a quadratic harmonic morphism, then, up to an isometry of R^3, φ is the composition of an orthogonal projection $\pi: R_r^3\to R_t^2$ (either $t=r$ or $t=r-1$) followed by a quadratic harmonic morphism $\varphi: R_t^2\to R_s^2$.

In the process of proof, we can describe the forms of these quadratic harmonic morphisms.

Theorem (2.1)′ Suppose that $\varphi: R_r^3\to R_s^2$ is a quadratic harmonic morphism with

$\varphi(X)=(X^tAX, X^tBX)$.

(i) If $r+s$ is even, then with respect to suitable coordinates in R_r^3, φ assumes the normal form

$$\varphi(X)=\left(X^t\begin{pmatrix}\varepsilon_1'\lambda & 0 & 0\\ 0 & \varepsilon_2'\lambda & 0\\ 0 & 0 & 0\end{pmatrix}X, X^t\begin{pmatrix}0 & \pm\lambda & 0\\ \pm\lambda & 0 & 0\\ 0 & 0 & 0\end{pmatrix}X\right)$$

where λ is the positive eigenvalue of A; $\varepsilon_i'=+1$ or -1, which satisfies $\varepsilon_1\varepsilon_1'+\varepsilon_2\varepsilon_2'=0$, where ε_i as previous stated.

(ii) If $r+s$ is odd, then with respect to suitable coordinates in R_r^3, φ assumes the normal form

$$\varphi(X)=\left(X^t\begin{pmatrix}0 & 0 & 0\\ 0 & \varepsilon_2'\lambda & 0\\ 0 & 0 & \varepsilon_3'\lambda\end{pmatrix}X, X^t\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \pm\lambda\\ 0 & \pm\lambda & 0\end{pmatrix}X\right),$$

where $\varepsilon_i'=+1$ or -1, which satisfies $\varepsilon_2\varepsilon_2'+\varepsilon_3\varepsilon_3'=0$.

Remark 2.1 Since it is restricted by the condition (3) in Proposition 1.2, if $r=0$ and $s=1$ (or $r=3$ and $s=1$) then mapping $\varphi: R_r^3\to R_s^2$ is no longer an quadratic harmonic morphism. Of course, these cases are excluded in Theorem 2.1.

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