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Finite Non-solvable Groups with the Given Length of Conjugacy Classes*

具有给定共轭类长的有限非可解群

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Abstract: Let G be a finite non-solvable group and $Z(G) = 1$. If the non-central conjugacy-class-lengths of G are pq, pr^2, qr^2, then $G \cong A_5$. If the non-central conjugacy-class-lengths of G are $15, 5p, 15p, 5p^2, 3p^3$, then $G \cong S_5$.

Key words: finite group, non-solvable group, conjugacy class, graph

摘要：设 G 是有限非可解群且 $Z(G)=1$. 如果 G 的非中心共轭类长为 pq, pr^2, qr^2，那么 G 同构于 5 次交错群 A_5；如果 G 的非中心共轭类长为 $15, 5p, 15p, 5p^2, 3p^3$，那么 G 同构于 5 次对称群 S_5.

关键词：有限群 非可解群 共轭类 图

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1 Introduction

Using some quantities of conjugacy classes of groups, many authors have described the structure of a finite group, in Reference [1], Bertram, Herzog and Mann obtained the following graph $\Gamma(G)$ to a group G. The vertices of $\Gamma(G)$ are represented by the non-central conjugacy classes of G, and connect two vertices D and C with an edge if $|D|$ and $|C|$ have a common prime divisor.

In the recent years, by studying the properties of $\Gamma(G)$, many authors have obtained some interesting results[1~3]. This idea clearly leads to a reassuring result: G is Abelian if and only if $\Gamma(G)$ has no vertices. In Reference [2] M. Fang and P. Zhang obtained the complete list of all G such that G is a non-Abelian group with $\Gamma(G)$ containing no triangles.

One of the questions that were studied extensively is what can be said about the structure of G if G is a non-Abelian group with $\Gamma(G)$ containing some triangles. Answers in many cases were given[1,3]. Our main aim in this article is to prove that if G is a finite non-solvable group and $Z(G) = 1$. Then the alternating group of degree 5, A_5, is the unique group G such that G has the non-central conjugacy-class-lengths pq, pr^2, qr^2. In this case, $\Gamma(G)$ is a complete graph containing exactly four triangles. For the non-central conjugacy-class-lengths $15, 5p, 15p, 5p^2, 3p^3$ of G, we prove that G is the symmetric group S_5, and $\Gamma(S_5)$ is a complete graph containing exactly twenty triangles.

Let $\pi(G) = \{p \mid p \text{ is a prime and } p \text{ devides } |G|\}$, $\pi_c(G) = \{p \mid p \text{ is a prime and } G \text{ has a conjugacy class } C \text{ such that } p \text{ devides } |C|\}$, $N(G) = \{n \mid G \text{ has a non-central conjugacy class } C \text{ such that } |C| = n\}$. Clearly, $\pi_c(G) \subseteq \pi(G)$.

In this paper, all groups mentioned are assumed to be finite and p, q, r are distinct primes.

2 Preliminaries

In order to prove our results, we need the following lemmas, some of which are well known.

Lemma 1[3,5] If p is a prime, then p doesn't

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devide n for each $n \in N(G)$ if and only if G has the Sylow p-subgroup in its center $Z(G)$.

Lemma 2[4] Let G be a group with $Z(G) = 1$. Then $\pi(G) = \pi_c(G)$.

Lemma 3[4] If G has two elements x and y such that $xy = yx$ and $(|x|, |y|) = 1$. Then $C_G(xy) = C_G(x) \cap C_G(y)$.

Lemma 4[3,5] If 4 doesn't devide n for each $n \in N(G)$, then G is solvable.

Lemma 5[3] If $N \trianglelefteq G$, then $|x^N|$ devides $|x^G|$ for all $x \in N$.

Lemma 6 If $Z(G) = 1$ and $N(G) = \{pq, pr^2, qr^2\}$, then $|G| = r^2pq$.

Proof Since $Z(G) = 1$, we have $\pi(G) = \{r, p, q\}$ from Lemma 2 and so $|G| = r^\alpha p^\beta q^\gamma$. Clearly, $\alpha \geqslant 2, \beta \geqslant 1$, and $\gamma \geqslant 1$.

We will prove that $\alpha = 2, \beta = \gamma = 1$. Let $R \in Syl_r(G)$ and $1 \neq x \in Z(R), R \subseteq C_G(x)$. Hence, the length of the conjugacy class x^G is $|x^G| = |G:C_G(x)|$, which is a divisor of $|G:R|$. Thus, $|x^G| = pq$, $|C_G(x)| = r^\alpha p^{\beta-1} q^{\gamma-1}$. Assume $\beta > 1$. Then $p \mid |C_G(x)|$. Let $P^* \in Syl_p(C_G(x))$. Then there exists a Sylow p-subgroup P of G such that P^* is normal in P since $|P:P^*| = p$. Further, $P^* \cap Z(P) \neq 1$. Taking $1 \neq y \in P^* \cap Z(P)$. In the same way, we get $|C_G(y)| = r^{\alpha-2} p^\beta q^{\gamma-1}$. Since $y \in C_G(x), xy = yx$, and $(|x|, |y|) = 1$, we obtain $C_G(xy) = C_G(x) \cap C_G(y)$ by Lemma 3. Hence, $|C_G(xy)| \leqslant (|C_G(x)|, |C_G(y)|) = r^{\alpha-2} p^{\beta-1} q^{\gamma-1}$. It follows that $|(xy)^G| = |G| / |C_G(xy)| = r^2pqs$ for some integer s, contradicting $N(G) = \{pq, pr^2, qr^2\}$. So $\beta = 1$. Similarly, $\gamma = 1$.

Next we claim $\alpha = 2$. Suppose not, let $\alpha > 2$ and $P \in Syl_p(G)$. Taking $1 \neq x \in Z(P)$, then $P \subseteq C_G(x)$. Hence, $|x^G| = |G:C_G(x)|$, which is a divisor of $|G:P|$. Thus, $|x^G| = r^2q$, $|C_G(x)| = r^{\alpha-2}p$. Since $\alpha > 2$, we have that r devides $|C_G(x)|$. Let $R^{**} \in Syl_r(C_G(x))$. Then there exists a r-subgroup R^* of R, a Sylow r-subgroup of G such that R^* is normal in R with $R^{**} \triangleleft R^* \triangleleft R$. Furthermore, $R^{**} \cap Z(R^*) \neq 1$. Taking $1 \neq y \in R^{**} \cap Z(R^*)$ and so $R^* \leqslant C_G(y)$, $|y^G| = |G:C_G(y)|$ is a divisor of $|G:R^*| = rpq$. We get $|y^G| = pq$, $|C_G(y)| = r^\alpha$, and $|C_G(xy)| \leqslant (|C_G(x)|, |C_G(y)|) = r^{\alpha-2}$. It follows that $|(xy)^G| = |G| / |C_G(xy)| = r^2pqt$ for the integer t, it is a contradiction again. Furthermore, $|G| = r^2pq$, as desired.

By similar procedures, we have

Lemma 7 If $Z(G) = 1$ and $N(G) = \{pq, pqr, qr, qr^2, pr^3\}$, then $|G| = r^3pq$.

3 Main results

By the above several lemmas, we prove the main results as follows.

Theorem 1 If G is a non-solvable group with $Z(G) = 1$ and $N(G) = \{pq, pr^2, qr^2\}$, then $G \cong A_5$.

Proof Since $Z(G) = 1$ and $N(G) = \{pq, pr^2, qr^2\}$, we have $|G| = r^2pq$ from Lemma 6. It follows that $G \cong A_5$ since G is non-solvable.

Corollary 2[6] Let G be a group. If $Z(G) = 1$ and $N(G) = N(A_5)$, then G is isomorphic to A_5.

Proof Since $Z(G) = 1$ and $N(A_5) = \{12, 15, 20\}$, $|G| = 60$ from Lemma 6. If G is solvable, then there is a proper normal subgroup $H > 1$ of G. We may chose H such that H is a minimal normal subgroup of G. It follows that $|H| \in \{2, 3, 4, 5\}$.

On the other hand, by Lemma 5 H has the following class equation:

$$|H| = 1 + 12a + 15b + 20c.$$

Where $a \geqslant 0, b \geqslant 0, c \geqslant 0$ are integers. It is clear that there is no solutions of the class equation of H, and it is a contradiction. Hence we may assume G is non-solvable, and the conclusion follows from the Theorem 1.

By similar procedures, we have

Corollary 3 There are no group G such that $Z(G) = 1$ and $N(G) = \{12, 3p, 4p\}$, where $p > 5, p \not\equiv 1 \pmod{12}$.

Theorem 4 If G is a non-solvable group such that $Z(G) = 1$ and $N(G) = \{15, 5p, 15p, 5p^2, 3p^3\}$ with $p \notin \{3, 5\}$, then $p = 2$, and G is the symmetric group S_5.

Proof Since G is non-solvable, $p = 2$ from Lemma 4. For a prime s with $s \notin \pi = \{2, 3, 5\}$, from Lemma 1 that G has the Sylow s-subgroup in its center. Let H be a π'-Hall subgroup of G, by the Schur-Zassenhaus theorem, H has a complement

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当 $V = 0$ 时，

有 $\frac{b}{2}y^2(f(x) - b) = 0, cx + by = 0, \frac{c}{b}y + z = 0$，又 $f(x) - b > 0$，从而 $x = y = z = 0$，因而 $V(t,x,y,z)$ 是定正的. 由于 $\inf g(y) > \frac{c}{b}$，则存在 $\varepsilon > 0$ 使 $g(y) - \frac{c}{b} > \varepsilon$.

当 $y \neq 0$ 时，$V(t,x,y,z) > \frac{c}{2}\varepsilon y^2 > 0$；

当 $y = 0, x^2 + z^2 \neq 0$ 时，$V(t,x,y,z) = \frac{c}{2}x^2 + \frac{b}{2}z^2 > 0$，故 $V(t,x,y,z)$ 是无限大正定函数.

又 $V(t,x,y,z) \leqslant (cx+by)^2 + (b+1)\int_0^y[f(x) - b]\eta d\eta + b(\frac{c}{b}y + z)^2 + (c+1)\int_0^y[g(\eta) - \frac{c}{b}]\eta d\eta$，

且上式右端在 $(0,0,0)$ 处取值为 0，因而 $V(t,x,y,z)$ 有无穷小上界.

$\dot{V}_{(2.2)} = -c[f(x) - b]y^2 - [bg(y) - c]z^2 + \int_0^y bf'(x)y\eta d\eta + (cy + bz)e(t,x,y,z) \leqslant -c\varepsilon y^2 + 4(1 + \frac{c+1}{b} + \frac{b+1}{c})(2 + 2b + \frac{2}{c\varepsilon})\tilde{e}(t)$，

令 $\eta(t) = 4(1 + \frac{c+1}{b} + \frac{b+1}{c})(2 + 2b + \frac{2}{c\varepsilon})\tilde{e}(t)$，

由条件 $\int_0^\infty \tilde{e}(t)dt = E < +\infty$ 知

$$\int_0^\infty \eta(t)dt < +\infty,$$

由引理 1 知(2.2)的零解是全局渐近稳定的.

定理 2 证毕.

3 算例

例 1 考虑三阶非自治系统 $\dddot{x} + (\frac{1}{b} + 1 + e^{\dot{x}})\ddot{x} + b\dot{x} + \frac{e^x - e^{-x}}{2} = \frac{1}{1+t^2}(y+z)$ 的全局渐近稳定性.

解 这里 $f(x) = \frac{e^x - e^{-x}}{2}, f'(x) = \frac{e^x + e^{-x}}{2} \geqslant 1 > 0, g(y) = \frac{1}{b} + 1 + e^y \geqslant \frac{1}{b} + 1$，取 $a = \frac{1}{b} + 1$，则 $\inf g(y) \geqslant a, \inf f'(x) \leqslant 1 + b = (1 + \frac{1}{b})b = ab$，又 $|e(t,x,y,z)| \leqslant \frac{1}{1+t^2}(|y| + |z|)$，$\int_0^{+\infty}\frac{1}{1+t^2}dt = \frac{\pi}{2}$ 收敛，由定理 1 可知该系统是全局渐近稳定的.

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subgroup K in G such that $G = H \times K$ with $H \leqslant Z(G)$ and so $H = 1, K = G$. It is easy to see that $|G| = 120$ from Lemma 7 and so $G \cong S_5$ or $Z_2 \times A_5$, the direct product of a cyclic group of order 2 and A_5, or $SL(2,5)$ since G is non-solvable[7]. Note that $Z(Z_2 \times A_5) \cong Z_2$ and $Z(SL(2,5)) \cong Z_2$, we have $G \cong S_5$.

According to Theorem 4, for group S_5 with $N(S_5) = \{10,15,20,24,30\}$, we have the following result.

Corollary 5 If G is a non-solvable group such that $Z(G) = 1$ and $N(G) = N(S_5)$, then $G \cong S_5$.

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