广西科学 Guangxi Sciences 2006，13(2)：90～92

The Confirmation of Two Conjectures about the Difference Equation $x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}$*

差分方程 $x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}$ 两个猜想的证明

WANG Qi[1,2], ZENG Fan-ping[1], ZHANG Geng-rong[1]

王 琦，曾凡平，张更容

(1. College of Mathematics and Information Sciences, Guangxi University, Nanning, Guangxi, 530004, China; 2. Department of Information and Computer Science, Guangxi University of Technology, Liuzhou, Guangxi, 545006, China)

(1. 广西大学数学与信息科学学院，广西南宁 530004；2. 广西工学院信息与计算机科学系，广西柳州 545006)

Abstract: Every non-negative solution of the difference equation $x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}$, $n=0,1,\cdots$ where $C\in(0,+\infty)$, is bounded.

Key words: difference equation, periodic solution, positive solution, boundedness.

摘要：证明差分方程 $x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}$，$n=0,1,\cdots$ 当 $C\in(0,+\infty)$ 时每个非负解是有界的.

关键词：差分方程　周期解　正解　界性

中图法分类号：O175　　**文献标识码**：A　　**文章编号**：1005-9164(2006)02-0090-03

1 Introduction

In this note, we consider the difference equation

$$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}},\quad n=0,1,\cdots \tag{1.1}$$

where $\alpha,\beta,\gamma,A,B,C\in[0,\infty)$ with $\alpha+\beta+\gamma$, $A+B+C\in(0,\infty)$, and the initial conditions x_{-1}, x_0 are arbitrary non-negative real numbers such that the denominator of Equation (1.1) is never zero. The following conjectures in the reference [1] are introduced.

Conjecture 1[1] Assume that $C\in(0,\infty)$. Then every solution of Equation (1.1) is bounded.

Conjecture 2[1] Assume that $\alpha,\beta,A,B\in(0,\infty)$. Show that every positive solution of each of the following two equations is bounded.

$$x_{n+1}=\frac{\alpha+\beta x_n+x_{n-1}}{A+x_{n-1}},n=0,1,\cdots \tag{1.2}$$

$$x_{n+1}=\frac{\alpha+\beta x_n+x_{n-1}}{Bx_n+x_{n-1}},n=0,1,\cdots \tag{1.3}$$

It is obvious that when Conjecture 1 holds, Conjecture 2 holds naturally. Hence, it is sufficient to prove conjecture 1.

The aim of this paper is to confirm the above conjectures, namely,

Theorem Assume $C\in(0,\infty)$. Then every non-negative solution of Equation (1.1) is bounded.

2 Main result

Firstly, we prove the following lemmas which are the special cases of Equation (1.1) and are interesting in their own rights.

Lemma 2.1 Assume $\gamma=B=0$, $\beta=C=1$, $A>0$. In this case, Equation (1.1) reduces to

收稿日期：2005-08-02

修回日期：2005-10-11

作者简介：王 琦 (1978-)，男，山东菏泽人，主要从事动力系统研究。

* Supported by NSFC (10361001, 10226014), Guangxi Science Foundation (0249002, 0007002) and Science Foundation of Guangxi University (X061022).

$$x_{n+1}=\frac{\alpha+x_n}{A+x_{n-1}},\quad n=0,1,\cdots \tag{2.1}$$

Then every non-negative solution of Equation (2.1) is bounded.

Proof Let $\{x_n\}_{n=-1}^{\infty}$ be a non-negative solution of Equation (2.1). Suppose, for the sake of contradiction, that $\{x_n\}_{n=-1}^{\infty}$ is not bounded, then there exists a subsequence $\{x_{n_i}\}_{i=0}^{\infty}$ of $\{x_n\}_{n=-1}^{\infty}$ with $\lim_{i\to\infty} x_{n_i+1}=\infty$ and $x_{n_i+1}>x_m$ for all $-1\leqslant m<n_i+1$ and all $i=0,1,2,\cdots$. Equation (2.1) implies that $(A+x_{n_i-1})x_{n_i+1}=\alpha+x_{n_i}$, note that $A>0$, we have $\lim_{i\to\infty}x_{n_i}=\infty$. Similarly, we get $\lim_{i\to\infty}x_{n_i-1}=\infty$. Hence for i sufficiently large, we have $0<x_{n_i+1}-x_{n_i}=\frac{\alpha+(1-A-x_{n_i-1})x_{n_i}}{A+x_{n_i-1}}<0$, which is a contradiction. Hence $\{x_n\}_{n=-1}^{\infty}$ is bounded. The proof is completed.

Lemma 2.2 Assume $\gamma=B=A=0,\beta=C=1,\alpha>0$. In this case, Equation (1.1) reduces to

$$x_{n+1}=\frac{\alpha+x_n}{x_{n-1}},\quad n=0,1,\cdots \tag{2.2}$$

Then every non-negative solution of Equation (2.2) is bounded.

Proof Equation (2.2) is a special case of $x_{n+1}=\frac{\alpha+\beta x_n}{Cx_{n-1}}$ — Lyness' Equation (see the reference [2]). Hence it possesses the invariant

$$I_n=(\alpha+x_{n-1}+x_n)(1+\frac{1}{x_{n-1}})(1+\frac{1}{x_n})=$$

Constant for all $n=0,1,2,\cdots$ The proof is completed.

Lemma 2.3 Assume $\gamma=B=A=\alpha=0,\beta=C=1$. In this case, Equation (1.1) reduces to

$$x_{n+1}=\frac{x_n}{x_{n-1}},\quad n=0,1,\cdots \tag{2.3}$$

Then every non-negative solution of Equation (2.3) is bounded.

Proof In view of Equation (2.3), we get $x_{n+1}=1/x_{n-2}=1/(1/x_{n-5})=x_{n-5}$ for $n=4,5,\cdots$. It is easy to see that every positive solution of Equation (2.3) is periodic with period six. Hence every solution of Equation (2.3) is bounded. The proof is completed.

Now, we prove the main result of this paper.

Proof of Theorem In order to prove the theorem, we consider the following cases of Equation (1.1).

Case 1 $B>0,A>0$. Let $\{x_n\}_{n=-1}^{\infty}$ be a non-negative solution of Equation (1.1). Then $x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}\leqslant\frac{\alpha}{A}+\frac{\beta}{B}+\frac{\gamma}{C}$, for all $n\geqslant 0$, from which we reach the conclusion.

Case 2 $B>0,A=0,\alpha=0$. In this case, Equation (1.1) reduces to

$$x_{n+1}=\frac{\beta x_n+\gamma x_{n-1}}{Bx_n+Cx_{n-1}},\quad n=0,1,\cdots \tag{2.4}$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a non-negative solution of Equation (2.4). Then $x_{n+1}=\frac{\beta x_n+\gamma x_{n-1}}{Bx_n+Cx_{n-1}}\leqslant\frac{\beta}{B}+\frac{\gamma}{C}$ for all $n\geqslant 0$, from which we reach the conclusion.

Case 3 $B>0,A=0,\alpha>0,\gamma>0$. In this case, Equation (1.1) reduces to

$$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{Bx_n+Cx_{n-1}},\quad n=0,1,\cdots \tag{2.5}$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a non-negative solution of Equation (2.5). Suppose, for the sake of contradiction, that $\{x_n\}_{n=-1}^{\infty}$ is not bounded, then there exists a subsequence $\{x_{n_i}\}_{n=0}^{\infty}$ of $\{x_n\}_{n=-1}^{\infty}$ with $\lim_{i\to\infty}x_{n_i+1}=\infty$ and $x_{n_i+1}>x_m$ for all $-1\leqslant m<n_i+1$ and all $i=0,1,\cdots$.

We claim that $\lim_{i\to\infty}x_{n_i}=\lim_{i\to\infty}x_{n_i-1}=0$. Firstly, we prove that $\{x_{n_i}\}_{i=0}^{\infty}$ is bounded. Suppose, for the sake of contradiction, that $\{x_{n_i}\}_{i=0}^{\infty}$ is not bounded, assume, without loss of generality, that $\lim_{i\to\infty}x_{n_i}=\infty$. For sufficiently large i, we have $0<x_{n_i+1}-x_{n_i}=\frac{\alpha+(\beta-Bx_{n_i})x_{n_i}+(\gamma-Cx_{n_i})x_{n_i-1}}{Bx_{n_i}+Cx_{n_i-1}}<0$, which is a contradiction. Hence $\{x_{n_i}\}_{i=0}^{\infty}$ is bounded. Now it is in the position to complete the proof of the claim. Equation (2.5) implies that $0\leqslant x_{n_i-1}=\frac{(\alpha+\beta x_{n_i})/x_{n_i+1}-Bx_{n_i}}{C-\gamma/x_{n_i+1}}$. Since $\lim_{i\to\infty}x_{n_i+1}=\infty$ and $\{x_{n_i}\}_{i=0}^{\infty}$ is bounded, it is obvious that $\lim_{i\to\infty}x_{n_i}=\lim_{i\to\infty}x_{n_i-1}=0$.

From Equation (2.5) we get $x_{n_i-2}=\frac{\alpha+\beta x_{n_i-1}-Bx_{n_i}x_{n_i-1}}{Cx_{n_i}-\gamma}$. It is obvious that $\lim_{i\to\infty}x_{n_i-2}=-\frac{\alpha}{\gamma}<0$, which is a contradiction. Hence $\{x_n\}_{n=-1}^{\infty}$ is bounded.

Case 4 $B>0,A=0,\alpha>0,\gamma=0,\beta>0$. In this case, Equation (1.1) reduces to

$$x_{n+1}=\frac{\alpha+\beta x_n}{Bx_n+Cx_{n-1}},\quad n=0,1,\cdots \tag{2.6}$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a non-negative solution of Equation

(2.6). Suppose, for the sake of contradiction, that $\{x_n\}_{n=-1}^{\infty}$ is not bounded, then there exists a subsequence $\{x_{n_i+1}\}_{i=0}^{\infty}$ of $\{x_n\}_{n=-1}^{\infty}$ with $\lim_{i\to\infty} x_{n_i+1} = \infty$ and $x_{n_i+1} > x_m$ for all $-1 \leqslant m < n_i + 1$. It is similar to the proof of Lemma 2.6 that $\lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} x_{n_i-1} = 0$.

From Equation(2.6) we get $x_{n_i-2} = (\alpha + \beta x_{n_i-1} - Bx_{n_i}x_{n_i-1})/Cx_{n_i}$ and $x_{n_i-3} = (\alpha + \beta x_{n_i-2} - Bx_{n_i-1}x_{n_i-2})/Cx_{n_i-1}$. Clearly $\lim_{i\to\infty} x_{n_i-2} = \infty$, which implies that $\lim_{i\to\infty} x_{n_i-3} = \infty$.

Since $x_{n+1} - x_{n-3} = (\alpha + \beta x_n)/(Bx_n + Cx_{n-1}) - x_{n-3} = (\alpha + \beta x_n)/(Bx_n + C\dfrac{\alpha + \beta x_{n-2}}{Bx_{n-2} + Cx_{n-3}}) - x_{n-3} = ((\alpha + \beta x_n)(Bx_{n-2} + Cx_{n-3}))/(Bx_n \cdot (Bx_{n-2} + Cx_{n-3}) + C(\alpha + \beta x_{n-2})) - x_{n-3} = ((\alpha + \beta x_n)(Bx_{n-2} + Cx_{n-3}) - Bx_{n-3}x_n(Bx_{n-2} + Cx_{n-3}) - Cx_{n-3}(\alpha + \beta x_{n-2}))/(Bx_n(Bx_{n-2} + Cx_{n-3}) + C(\alpha + \beta x_{n-2})) = ((\alpha B - \beta Cx_{n-3})x_{n-2} + x_n(\beta - Bx_{n-3})(Bx_{n-2} + Cx_{n-3}))/(Bx_n(Bx_{n-2} + Cx_{n-3}) + C(\alpha + \beta x_{n-2}))$,

for sufficiently large i, we have $0 < x_{n_i+1} - x_{n_i-3} = ((\alpha B - \beta Cx_{n_i-3})x_{n_i-2} + x_{n_i}(\beta - Bx_{n_i-3})(Bx_{n_i-2} + Cx_{n_i-3}))/(Bx_{n_i}(Bx_{n_i-2} + Cx_{n_i-3}) + C(\alpha + \beta x_{n_i-2})) < 0$, which is a contradiction. Hence $\{x_n\}_{n=-1}^{\infty}$ is bounded.

Case 5 $B > 0, A = 0, \alpha > 0, \gamma = 0, \beta = 0$. In this case, Equation(1.1) reduces to

$$x_{n+1} = \frac{\alpha}{Bx_n + Cx_{n-1}}, n = 0,1,\cdots \qquad (2.7)$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a non-negative solution of Equation (2.7). Suppose, for the sake of contradiction, that $\{x_n\}_{n=-1}^{\infty}$ is not bounded, then there exists $k > 4$ such that $x_k > \max\{x_n : -1 \leqslant n < k\}$. Set $x_l = \min\{x_i : 1 \leqslant i < k\}$. Then

$$x_k = \frac{\alpha}{Bx_{k-1} + Cx_{k-2}} < \frac{\alpha}{B+C}\frac{1}{x_l}, \frac{\alpha}{B+C}\frac{1}{x_k} < \frac{\alpha}{Bx_{l-1} + Cx_{l-2}} = x_l,$$

from which we get $\dfrac{\alpha}{B+C} < \dfrac{\alpha}{B+C}$. It is a contradiction. Hence $\{x_n\}_{n=-1}^{\infty}$ is bounded.

Case 6 $B = 0, A > 0$. In this case, Equation (1.1) reduces to

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Cx_{n-1}}, n = 0,1,\cdots \qquad (2.8)$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a non-negative solution of Equation (2.8). Then $x_{n+1} = \dfrac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Cx_{n-1}} < \dfrac{\alpha}{A} + \dfrac{\gamma}{C} + \dfrac{\beta x_n}{A + Cx_{n-1}}$, $x_{n+2} < \dfrac{\alpha}{A} + \dfrac{\gamma}{C} + \dfrac{\beta}{A + Cx_n}(\dfrac{\alpha}{A} + \dfrac{\gamma}{C} + \dfrac{\beta x_n}{A + Cx_{n-1}}) < \dfrac{\alpha}{A} + \dfrac{\gamma}{C} + \dfrac{\beta}{A}(\dfrac{\alpha}{A} + \dfrac{\gamma}{C}) + \dfrac{\beta^2}{AC}$ for all $n \geqslant 0$, from which we reach the conclusion.

Case 7 $B = 0, A = 0, \beta > 0$. In this case, Equation(1.1) reduces to

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{Cx_{n-1}}, n = 0,1,\cdots \qquad (2.9)$$

Let $\{x_n\}_{n=-1}^{\infty}$ be a non-negative solution of Equation (2.9). It is obvious that $x_n \geqslant \dfrac{\gamma}{C}$ for all $n \geqslant 1$.

With a transformation $x_n = \dfrac{\beta}{C}y_n + \dfrac{\gamma}{C}$, Equation (2.9) can be transformed to

$$y_{n+1} = \frac{\dfrac{\alpha C + \beta\gamma}{\beta^2} + y_n}{\dfrac{\gamma}{\beta} + y_{n-1}}, n = 0,1,\cdots \qquad (2.10)$$

From Lemmas 2.1, 2.2 and 2.3, we get that every solution of Equation (2.10) is bounded, hence every solution of Equation(2.9) is bounded.

Case 8 $B = 0, A = 0, \beta = 0, \gamma = 0$. In this case, Equation(1.1) reduces to

$$x_{n+1} = \frac{\alpha}{Cx_{n-1}}, n = 0,1,\cdots \qquad (2.11)$$

In view of Equation(2.11), we get $x_{n+1} = \alpha/Cx_{n-1} = \alpha/C(\alpha/Cx_{n-3}) = x_{n-3}$ for $n = 2,3,\cdots$. It is easy to see that every solution of Equation (2.11) is periodic within period-four. Hence every solution of Equation (2.11) is bounded.

Case 9 $B = 0, A = 0, \beta = 0, \gamma > 0$. In this case, Equation(1.1) reduces to

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{Cx_{n-1}}, n = 0,1,\cdots \qquad (2.12)$$

It is obvious that $x_{n+1} \geqslant \dfrac{\gamma}{C}$ for all $n \geqslant 1$. Hence $x_{n+1} = \dfrac{\alpha + \gamma x_{n-1}}{Cx_{n-1}} \leqslant \dfrac{\alpha}{C\dfrac{\gamma}{C}} + \dfrac{\gamma}{C} = \dfrac{\alpha}{\gamma} + \dfrac{\gamma}{C}$, for all $n \geqslant 1$, from which we reach the conclusion.

From the above analysis, we complete the proof of theorem.

References:

[1] CAMOUZIS E, CHATTERJEE E, LADAS G, et al. Open problems and Conjectures[J]. J Differ Equ Appl, 2004, 10(12): 1119-1127.

[2] KULENOVIC M R S, LADAS G. Dynamics of second order rational difference equations with open problems and conjectures[M]. Boca Raton: Chapman & Hall/CRC Press, 2001.

(责任编辑:邓大玉)