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Global Asymptotic Stability of Two Families of Nonlinear Difference Equations*

两类非线性差分方程的全局渐近稳定性

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Abstract: Two families of difference equations are discussed. They are the form

$$x_{n+1}=\frac{\sum_{i\in Z_k-\{j,s,t\}}x_{n-i}+x_{n-t}^r+x_{n-j}x_{n-s}^m+A}{\sum_{i\in Z_k-\{j,s,t\}}x_{n-i}+x_{n-s}^m+x_{n-j}x_{n-t}^r+A},n=0,1,\cdots,$$

where $k\in\{2,3,\cdots\}$, $j,s,t\in \mathbf{Z}_k\equiv\{0,1,\cdots,k\}$ with $s\neq t$ and $j\notin\{s,t\}$, $A,r,m\in[0,+\infty)$ and the initial values $x_{-k},x_{-k+1},\cdots,x_0\in(0,+\infty)$, and the form

$$x_{n+1}=\frac{\sum_{i\in Z_k-\{j_0,j_1,\cdots,j_s\}}x_{n-i}+x_{n-j_0}x_{n-j_1}\cdots x_{n-j_s}+1}{\sum_{i\in Z_k-\{j_0,j_1,\cdots,j_{s-1}\}}x_{n-i}+x_{n-j_0}x_{n-j_1}\cdots x_{n-j_{s-1}}},n=0,1,\cdots,$$

where $k\in\{1,2,3,\cdots\}$, $1\leqslant s\leqslant k$, $\{j_0\cdots,j_s\}\subset \mathbf{Z}_k$ with $j_i\neq j_l$ for $i\neq l$ and the initial values $x_{-k},x_{-k+1},\cdots,x_0\in(0,+\infty)$. For these difference equations, it is proved that the unique equilibrium $\bar{x}=1$ is globally asymptotically stable, which includes the corresponding results of the references [3～5,7].

Key words: difference equation, equilibrium, global asymptotic stability

摘要:利用泛函分析方法证明差分方程

$$x_{n+1}=\frac{\sum_{i\in Z_k-\{j,s,t\}}x_{n-i}+x_{n-t}^r+x_{n-j}x_{n-s}^m+A}{\sum_{i\in Z_k-\{j,s,t\}}x_{n-i}+x_{n-s}^m+x_{n-j}x_{n-t}^r+A},n=0,1,\cdots,$$

其中 $k\in\{2,3,\cdots\}$, $j,s,t\in Z_k\equiv\{0,1,\cdots,k\}(s\neq t,j\notin\{s,t\})$, $A,r,m\in[0,+\infty)$ 且初始条件 $x_{-k},x_{-k+1},\cdots,x_0\in(0,+\infty)$, 和差分方程

$$x_{n+1}=\frac{\sum_{i\in Z_k-\{j_0,j_1,\cdots,j_s\}}x_{n-i}+x_{n-j_0}x_{n-j_1}\cdots x_{n-j_s}+1}{\sum_{i\in Z_k-\{j_0,j_1,\cdots,j_{s-1}\}}x_{n-i}+x_{n-j_0}x_{n-j_1}\cdots x_{n-j_{s-1}}},n=0,1,\cdots,$$

其中 $k\in\{1,2,3,\cdots\}$, $1\leqslant s\leqslant k$, $\{j_0,\cdots,j_s\}\subset Z_k(j_i\neq j_l$ 对 $i\neq l)$ 且初始条件 $x_{-k},x_{-k+1},\cdots,x_0\in(0,+\infty)$ 的唯一平衡点 $\bar{x}=1$ 是全局渐近稳定的. 该结果推广了文献[3～5,7]中相应的结果.

关键词:差分方程　平衡点　全局渐近稳定性

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1 Introduction

For some difference equations, although their forms (or expressions) look very simple, it is extremely difficult to understand the global behaviors of their solutions thoroughly. Some previous investigations on the qualitative behaviors of difference

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equations have been seen in references [1~5]).

In reference [3] Ladas put forward to investigate the global asymptotic stability of the following rational difference equation:

$$x_{n+1} = \frac{x_n + x_{n-1}x_{n-2}}{x_n x_{n-1} + x_{n-2}}, n = 0,1,\cdots \qquad \text{(E1)}$$

where the initial values $x_{-2}, x_{-1}, x_0 \in R_+ \equiv (0, +\infty)$.

In reference [4] Nesemann utilized the strong negative feedback property of reference [6] to study the following difference equation:

$$x_{n+1} = \frac{x_{n-1} + x_n x_{n-2}}{x_n x_{n-1} + x_{n-2}}, n = 0,1,\cdots \qquad \text{(E2)}$$

where the initial values $x_{-2}, x_{-1}, x_0 \in R_+$.

In reference [5] Li and Zhu studied the global asymptotic stability of the following nonlinear difference equation:

$$x_{n+1} = \frac{x_n x_{n-1}^r + x_{n-2}^r + A}{x_{n-1}^r + x_n x_{n-2}^r + A}, n = 0,1,\cdots \qquad \text{(E3)}$$

where $A, r \in [0, +\infty)$ and the initial values $x_{-2}, x_{-1}, x_0 \in R_+$.

Recently, Papaschinopoulos and Schinas[7] have investigated the global asymptotic stability of the following nonlinear difference equation:

$$x_{n+1} = \frac{\sum_{i \in Z_k - \{j-1,j\}} x_{n-i} + x_{n-j}x_{n-j+1} + 1}{\sum_{i \in Z_k} x_{n-i}}, n = 0, 1, \cdots \qquad \text{(E4)}$$

where $k \in \{1,2,3,\cdots\}$, $\{j, j-1\} \subset \mathbf{Z}_k \equiv \{0,1,\cdots,k\}$ and the initial values $x_{-k}, x_{-k+1}, \cdots, x_0 \in R_+$.

In this note, we consider the family of difference equations of the form

$$x_{n+1} = \frac{\sum_{i \in Z_k - \{j,s,t\}} x_{n-i} + x_{n-t}^r + x_{n-j}x_{n-s}^m + A}{\sum_{i \in Z_k - \{j,s,t\}} x_{n-i} + x_{n-s}^m + x_{n-j}x_{n-t}^r + A}, n = 0,1,\cdots \qquad (1)$$

where $k \in \{2,3,\cdots\}$, $j,s,t \in \mathbf{Z}_k$ with $s \neq t$ and $j \notin \{s,t\}$, $A, r, m \in [0, +\infty)$ and the initial values $x_{-k}, x_{-k+1}, \cdots, x_0 \in R_+$, and the family of difference equations of the form

$$x_{n+1} = \Big(\sum_{i \in Z_k - \{j_0, j_1, \cdots, j_s\}} x_{n-i} + x_{n-j_0} x_{n-j_1} \cdots x_{n-j_s} + 1\Big) \Big/ \Big(\sum_{i \in Z_k - \{j_0, j_1, \cdots, j_{s-1}\}} x_{n-i} + x_{n-j_0} x_{n-j_1} \cdots x_{n-j_{s-1}}\Big), n = 0, 1, \cdots \qquad (2)$$

where $k \in \{1,2,3,\cdots\}$, $1 \leqslant s \leqslant k$, $\{j_0 \cdots, j_s\} \subset \mathbf{Z}_k$ with $j_i \neq j_l$ for $i \neq l$ and the initial values $x_{-k}, x_{-k+1}, \cdots, x_0 \in R_+$.

It is easy to see that the positive equilibrium $\bar{x}$ of Equation (1) satisfies

$$\bar{x} = \frac{(k-2)\bar{x} + \bar{x}^r + \bar{x}^{m+1} + A}{(k-2)\bar{x} + \bar{x}^m + \bar{x}^{r+1} + A}$$

and the positive equilibrium $\bar{x}$ of Equation (2) satisfies

$$\bar{x} = \frac{\bar{x}^{s+1} + (k-s)\bar{x} + 1}{\bar{x}^s + (k-s+1)\bar{x}},$$

from which it can be seen that Equations (1) and (2) have the unique positive equilibrium $\bar{x} = 1$.

The following theorem is our main result, which includes the corresponding results of references [3~5, 7].

Theorem (i) Assume that $A, r, m \in [0, +\infty)$. Then the unique equilibrium $\bar{x} = 1$ of Equation (1) is globally asymptotically stable.

(ii) The unique equilibrium $\bar{x} = 1$ of Equation (2) is globally asymptotically stable.

2 Proof of the theorem

To prove the Theorem, we need the following lemmas.

Lemma 1 Let $k \in \{2,3,\cdots\}$ and $A, r, m \in [0, +\infty)$. If $(a,b,c,u_1,\cdots,u_{k-2}) \in R_+^{k+1} - \{(1,1,\cdots,1,1)\}$ and $\alpha = \max\{a,b,c,u_1,\cdots,u_{k-2},a^{-1},b^{-1},c^{-1},u_1^{-1},\cdots,u_{k-2}^{-1}\}$, then

$$\frac{1}{\alpha} < \frac{ab^m + c^r + u_1 + \cdots + u_{k-2} + A}{ac^r + b^m + u_1 + \cdots + u_{k-2} + A} < \alpha.$$

Proof Since $(a,b,c,u_1,\cdots,u_{k-2}) \in R_+^{k+1} - \{(1,1,\cdots,1,1)\}$ and $\alpha = \max\{a,b,c,u_1,\cdots,u_{k-2},a^{-1},b^{-1},c^{-1},u_1^{-1},\cdots,u_{k-2}^{-1}\}$, we have $\alpha > 1$ and either $\alpha \geqslant a > \frac{1}{\alpha}$ or $\alpha > a \geqslant \frac{1}{\alpha}$. Then

$$\begin{cases} ab^m + c^r < ab^m + \alpha ac^r, \\ ac^r + b^m < ac^r + \alpha ab^m. \end{cases} \qquad (3)$$

It follows from Formula (3) that

$$\frac{1}{\alpha} < \frac{ab^m + c^r + u_1 + \cdots + u_{k-2} + A}{ac^r + b^m + u_1 + \cdots + u_{k-2} + A} < \alpha.$$

Lemma 1 is proven.

Lemma 2 Let $k \in \{1,2,3,\cdots\}$ and $s \in \{1,\cdots,k\}$. If $(a_1,a_2,\cdots,a_s,b_0,\cdots,b_{k-s}) \in R_+^{k+1} - \{(1,1,\cdots,1,1)\}$ and $\alpha = \max\{a_1,a_2,\cdots,a_s,b_0,\cdots,b_{k-s},a_1^{-1},a_2^{-1},\cdots,a_s^{-1},b_0^{-1},\cdots,b_{k-s}^{-1}\}$, then

$$\frac{1}{\alpha} < \frac{a_1a_2\cdots a_s + b_0 + \cdots + b_{k-s} + 1}{a_1a_2\cdots a_{s-1} + a_s + b_0 + \cdots + b_{k-s}} < \alpha.$$

Proof Since $(a_1,a_2,\cdots,a_s,b_0,\cdots,b_{k-s}) \in R_+^{k+1} - \{(1,1,\cdots,1,1)\}$ and $\alpha = \max\{a_1,a_2,\cdots,a_s,b_0,\cdots,b_{k-s},a_1^{-1},a_2^{-1},\cdots,a_s^{-1},b_0^{-1},\cdots,b_{k-s}^{-1}\}$, we have $\alpha > 1$ and either $\alpha \geqslant a_s > \frac{1}{\alpha}$ or $\alpha > a_s \geqslant \frac{1}{\alpha}$. Then

$$\begin{cases} a_1 a_2 \cdots a_s + 1 < a_1 a_2 \cdots a_{s-1}\alpha + a_s \alpha, \\ a_1 a_2 \cdots a_{s-1} + a_s < a_1 a_2 \cdots a_s \alpha + \alpha. \end{cases} \quad (4)$$

It follows from Formula(4) that

$$\frac{1}{\alpha} < \frac{a_1 a_2 \cdots a_s + b_0 + \cdots + b_{k-s} + 1}{a_1 a_2 \cdots a_{s-1} + a_s + b_0 + \cdots + b_{k-s}} < \alpha.$$

Lemma 2 is proven.

Let ρ denote the part-metric on R_+^{k+1} (see reference [8]) which is defined by

$\rho(x, y) = -\log\min\{\frac{x_i}{y_i}, \frac{y_i}{x_i} | 0 \leqslant i \leqslant k\}$ for $x = (x_0, \cdots, x_k), y = (y_0, \cdots, y_k) \in R_+^{k+1}$.

It was shown by Thompson[8] that (R_+^{k+1}, ρ) is a complete metric space. In reference [9] Krause and Nussbaum proved that the distances indicated by the part-metric and by the Euclidean norm were equivalent on R_+^{k+1}.

Lemma 3[10] Let $T: R_+^{k+1} \to R_+^{k+1}$ be a continuous mapping with unique fixed point $x^* \in R_+^{k+1}$. Suppose that there exists some $l \geqslant 1$ such that for the part-metric ρ,

$$\rho(T^l x, x^*) < \rho(x, x^*) \text{ for all } x \neq x^*.$$

Then x^* is globally asymptotically stable.

Proof of theorem Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Equation (1) (or Equation (2)) with initial conditions $x_0, x_{-1}, \cdots, x_{-k} \in R_+$ such that $\{x_n\}_{n=-k}^{\infty}$ is not eventually equal to 1 since otherwise there is nothing to show. Denoted by $T: R_+^{k+1} \to R_+^{k+1}$ the mapping

$T(a_0, a_1, \cdots, a_k) = (a_1, a_2, \cdots, a_k, f(a_0, a_1, \cdots, a_k))$, where

$$f(a_0, a_1, \cdots, a_k) = \frac{\sum_{i \in Z_k - \{j,s,t\}} a_{k-i} + a_{k-t}^r + a_{k-j} a_{k-s}^m + A}{\sum_{i \in Z_k - \{j,s,t\}} a_{k-i} + a_{k-s}^m + a_{k-j} a_{k-t}^r + A}$$

(or

$$f(a_0, a_1, \cdots, a_k) = \frac{\sum_{i \in Z_k - \{j_0, j_1, \cdots, j_s\}} a_{k-i} + a_{k-j_0} a_{k-j_1} \cdots a_{k-j_s} + 1}{\sum_{i \in Z_k - \{j_0, j_1, \cdots, j_{s-1}\}} a_{k-i} + a_{k-j_0} a_{k-j_1} \cdots a_{k-j_{s-1}}}).$$

Then solution $\{x_n\}_{n=-k}^{\infty}$ of Equation (1) (or Equation (2)) is represented by the first component of the solution $\{y_n\}_{n=0}^{\infty}$ of the system $y_{n+1} = T y_n$ with initial condition $y_0 = (x_{-k}, \cdots, x_{-1}, x_0)$. It follows from Lemma 1 (or Lemma 2) that for all $n \geqslant 0$ the following inequalities hold:

$$x_{n+1} > \min\{x_n, x_{n-1}, \cdots, x_{n-k}, \frac{1}{x_n}, \frac{1}{x_{n-1}}, \cdots, \frac{1}{x_{n-k}}\},$$

$$x_{n+1} < \max\{x_n, x_{n-1}, \cdots, x_{n-k}, \frac{1}{x_n}, \frac{1}{x_{n-1}}, \cdots, \frac{1}{x_{n-k}}\}.$$

Thus, for $x^* = (1, 1, \cdots, 1)$ and the part-metric ρ we have $\rho(T^{k+1}(y_n), x^*) < \rho(y_n, x^*)$ for all $n \geqslant 0$. It follows from Lemma 3 that the positive equilibrium $\bar{x} = 1$ of Equation (1) (or Equation (2)) is globally asymptotically stable.

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