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Uniqueness of Solutions to a Degenerate Parabolic Equation

一种退化抛物方程广义解的惟一性

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Abstract: The uniqueness of solutions to the initial boundary value problems of a degenerate parabolic equation is proved by means of a regularizing technique based on elliptic operators.

Key words: parabolic equation, uniqueness, elliptic operator

摘要：借助正则化技术，基于椭圆算子证明一种退化抛物方程初边值问题广义解的惟一性.

关键词：抛物方程　惟一性　椭圆算子

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1 Introduction

This paper is concerned to the uniqueness of solutions to the following initial boundary value problems

$$\frac{\partial C(u)}{\partial t} = \Delta A(u) + \operatorname{div} \vec{B}(u), (x,t) \in Q_T, \tag{1.1}$$

$$A(u(x,t)) = 0, (x,t) \in \partial\Omega \times (0,T), \tag{1.2}$$

$$C(u(x,0)) = u_0(x), x \in \Omega. \tag{1.3}$$

Where Ω is a bounded domain with smooth boundary in R^N, $Q_T = \Omega \times (0,T)$, $A(s)$ and $C(s)$ are continuous differential function on R with $A'(s) \geqslant 0$ and $C'(s) \geqslant 0$, $A(s) = \Phi_1(C(s))$, $\vec{B}(s) = \vec{\Phi}_2(A(s))$ and $\Phi_1(s)$, $\vec{\Phi}_2(s)$ are locally Lipschite continuously functions and $u_0(x) \in L^\infty(\Omega)$.

Equation (1.1) is a kind of equation with double degeneracies. Generally, as $C'(s) \neq 0$, this is a typical parabolic-hyperbolic mixed equation, which degenerates when $A'(s) = 0$. Similarly, as $A'(s) \neq 0$, it is the elliptic-parabolic mixed type, which degenerates when $C'(s) = 0$. And its significance is not only due to theoretical research but also due to its physical background. In the case of $C(s) = s$, this kind of equations have been paid much attention by many pioneers. The discussion of existence and uniqueness of solutions can be found in many papers[1~5].

In this paper, we deal with the uniqueness of generalized solutions to the problems (1.1)～(1.3). The method used was inspired by some ideas of Brézis and Crandall[6]. We do not consider the Cauchy problem but the initial boundary value problem, so the regularizing procedure is achieved by means of an elliptic operator.

Definition 1.1 A function $u(x,t)$ is called a generalized solution of the initial boundary value problems (1.1)～(1.3), if $u \in L^\infty(Q_T)$, $A(u) \in L^2(0,T;H_0^1(\Omega))$, $C(u) \in BV(Q_T)$, and for any test function $\varphi \in C^\infty(\overline{Q_T})$ with $\varphi(x,t) = 0$ for $x \in \partial\Omega$ and for $t = T$, then the following integral equality holds

$$\iint_{Q_T} (C(u)\frac{\partial\varphi}{\partial t} + A(u)\Delta\varphi - \vec{B}(u) \cdot \nabla\varphi)\mathrm{d}x\mathrm{d}t + \int_\Omega u_0(x)\varphi(x,0)\mathrm{d}x = 0. \tag{1.4}$$

2 Results

Theorem 2.1 If u_1, u_2 are two generalized solutions of the initial boundary value problems (1.1)～(1.3), then $C(u_1) = C(u_2)$ a.e. in Q_T.

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Proof Since $u_1, u_2 \in L^\infty(Q_T)$ are two generalized solutions of the problems (1.1)~(1.3), by the definition of generalized solutions, for any test function $\varphi \in C^\infty(\overline{Q_T})$ with $\varphi(x,t) = 0$ for $x \in \partial\Omega$ and for $t = T$, we have

$$\iint_{Q_T} \overline{C}\frac{\partial\varphi}{\partial t}\mathrm{d}x\mathrm{d}t + \iint_{Q_T} \overline{A}\Delta\varphi\,\mathrm{d}x\mathrm{d}t - \iint_{Q_T} \vec{H}\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t = 0, \tag{2.1}$$

where

$$\overline{C} = C(u_1) - C(u_2), \overline{A} = A(u_1) - A(u_2), \vec{H} = \vec{B}(u_1) - \vec{B}(u_2).$$

We assert that $\overline{C} = 0$ a. e. in Q_T. From this equality one can choose special test functions φ.

For small $\lambda > 0$, we define the operator T_λ as the following

$$T_\lambda : L^2(\Omega) \to L^2(\Omega), f \mapsto u,$$

where $u = T_\lambda$ and f is determined uniquely by the Dirichlet problem

$$-\Delta u + \lambda u = f, x \in \Omega, \tag{2.2}$$

$$u = 0, x \in \partial\Omega.$$

It is easy to see that T_λ is a self-adjoint operator, i. e., for arbitrary $f, g \in L^2(\Omega)$,

$$\int_\Omega f T_\lambda g\mathrm{d}x = \int_\Omega g T_\lambda f\mathrm{d}x,$$

and from the L^2 theory for elliptic equations, we have

$$\|T_\lambda f\|_{L^2(\Omega)} \leqslant C_0 \|f\|_{L^2(\Omega)},$$

where C_0 is a constant depending only on N and Ω but independent of λ and f.

Replace φ by $T_\lambda\varphi$ in the formula (2.1), we get

$$\iint_{Q_T} \overline{C}\frac{\partial T_\lambda\varphi}{\partial t}\mathrm{d}x\mathrm{d}t + \iint_{Q_T} \overline{A}\Delta T_\lambda\varphi\mathrm{d}x\mathrm{d}t - \iint_{Q_T} \vec{H}\cdot\nabla T_\lambda\varphi\mathrm{d}x\mathrm{d}t = \iint_{Q_T} \overline{C}T_\lambda(\frac{\partial\varphi}{\partial t})\mathrm{d}x\mathrm{d}t + \iint_{Q_T} \overline{A}(\lambda T_\lambda\varphi - \varphi)\mathrm{d}x\mathrm{d}t + \iint_{Q_T} (\mathrm{div}\,\vec{H}) T_\lambda\varphi\mathrm{d}x\mathrm{d}t = \iint_{Q_T} \frac{\partial\varphi}{\partial t} T_\lambda\overline{C}\mathrm{d}x\mathrm{d}t + \iint_{Q_T} (\lambda T_\lambda\overline{A} - \overline{A})\varphi\,\mathrm{d}x\mathrm{d}t + \iint_{Q_T} T_\lambda(\mathrm{div}\,\vec{H})\varphi\,\mathrm{d}x\mathrm{d}t = \iint_{Q_T} -\frac{\partial T_\lambda\overline{C}}{\partial t}\varphi\,\mathrm{d}x\mathrm{d}t + \iint_{Q_T} (\lambda T_\lambda\overline{A} - \overline{A})\varphi\,\mathrm{d}x\mathrm{d}t + \iint_{Q_T} T_\lambda(\mathrm{div}\,\vec{H})\varphi\,\mathrm{d}x\mathrm{d}t = 0.$$

Owing to the arbitrariness of φ, we have that

$$\frac{\partial T_\lambda\overline{C}}{\partial t} = \lambda T_\lambda\overline{A} - \overline{A} + T_\lambda(\mathrm{div}\,\vec{H}) \tag{2.3}$$

in the sense of distribution.

Denote

$$g_\lambda(t) = \int_\Omega \overline{C}(x,t) T_\lambda\overline{C}(x,t)\mathrm{d}x,$$

we can prove that

$$\lim_{\lambda\to 0} g_\lambda(t) = 0, \quad a.e.\, t \in (0,T). \tag{2.4}$$

Once this is done, owing to the fact that

$$g_\lambda(t) = \int_\Omega \overline{C}(x,t) T_\lambda\overline{C}(x,t)\mathrm{d}x = \int_\Omega (\lambda T_\lambda\overline{C}(x,t) - \Delta T_\lambda\overline{C}(x,t)) T_\lambda\overline{C}(x,t)\mathrm{d}x = \lambda\|T_\lambda\overline{C}(.,t)\|_2^2 + \|\nabla T_\lambda\overline{C}(.,t)\|_2^2,$$

we have, for almost all $t \in (0,T)$,

$$\lim_{\lambda\to 0}\lambda T_\lambda\overline{C}(.,t) = 0, \lim_{\lambda\to 0}\nabla T_\lambda\overline{C}(.,t) = 0, \text{in } L^2(\Omega).$$

It follows that, in particular in the sense of distributions,

$$\lim_{\lambda\to 0}\lambda T_\lambda\overline{C}(.,t) = 0, \lim_{\lambda\to 0}\Delta T_\lambda\overline{C}(.,t) = 0, \text{in } L^2(\Omega),$$

and hence

$$\overline{C}(.,t) = \lim_{\lambda\to 0}(\lambda T_\lambda\overline{C}(.,t) - \Delta T_\lambda\overline{C}(.,t)) = 0, \text{in } L^2(\Omega).$$

This shows that $\overline{C} = C(u_1) - C(u_2) = 0$ in a. e. Q_T.

Now we turn to the proof of the formula (2.4). Let J_ε be the standard mollifier in t. From the formula (2.3) and the symmetry of T_λ, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega T_\lambda(J_\varepsilon\overline{C})J_\varepsilon\overline{C}\mathrm{d}x = 2\int_\Omega \frac{\partial}{\partial t}T_\lambda(J_\varepsilon\overline{C})J_\varepsilon\overline{C}\mathrm{d}x = 2\int_\Omega J_\varepsilon(\frac{\partial}{\partial t}T_\lambda\overline{C})J_\varepsilon\overline{C}\mathrm{d}x = 2\int_\Omega J_\varepsilon(\lambda T_\lambda\overline{A} - \overline{A} + T_\lambda(\mathrm{div}\,\vec{H}))J_\varepsilon\overline{C}\mathrm{d}x.$$

Integrating with respect to t give

$$\int_\Omega T_\lambda(J_\varepsilon\overline{C}(x,t))J_\varepsilon\overline{C}(x,t)\mathrm{d}x = \int_\Omega T_\lambda(J_\varepsilon\overline{C}(x,0))J_\varepsilon\overline{C}(x,0)\mathrm{d}x + 2\iint_{Q_t} J_\varepsilon(\lambda T_\lambda\overline{A} - \overline{A} + T_\lambda(\mathrm{div}\,\vec{H}))J_\varepsilon\overline{C}\mathrm{d}x\mathrm{d}s.$$

Let $\varepsilon \to 0$ and note $\overline{C}(x,0) = C(u_1(x,0)) - C(u_2(x,0)) = u_0(x) - u_0(x) = 0$, we have

$$g_\lambda(t) = 2\iint_{Q_t} (\lambda T_\lambda\overline{A} - \overline{A} + T_\lambda(\mathrm{div}\,\vec{H}))\overline{C}\mathrm{d}x\mathrm{d}s. \tag{2.5}$$

From $C(u_1), C(u_2) \in BV(Q_T)$, we know that $\overline{C}$ is bounded, $A(s), C(s)$ are continuously differentiable and $\Phi_1(s), \overline{\Phi}_2(s)$ are locally Lipschite continuous, there must exist two positive constants M_0 and M_1 such that the following estimate holds

$$|\overline{A}| \leqslant M_0|\overline{C}|, |\vec{H}| \leqslant M_1|\overline{A}|.$$

$\overline{C}$ and $\overline{A}$ have the same sign due to the assumption that $A'(s) \geqslant 0$ and $C'(s) \geqslant 0$, we get

$$\overline{C}\,\overline{A} = |\overline{C}||\overline{A}| \geqslant \frac{1}{M_0}|\overline{A}|^2.$$

By using of Schwartz's inequality and Young's inequality, we get

$$|\int_\Omega T_\lambda(\mathrm{div}\,\vec{H})\overline{C}\mathrm{d}x| = |\int_\Omega \mathrm{div}\,\vec{H}\cdot T_\lambda\overline{C}\mathrm{d}x| =$$

$$|\int_\Omega \vec{H}\cdot\nabla T_\lambda\overline{C}\mathrm{d}x| \leqslant (\int_\Omega |\vec{H}|^2\mathrm{d}x)^{1/2}\cdot (\int_\Omega |\nabla T_\lambda\overline{C}|^2\mathrm{d}x)^{1/2} \leqslant M_1(\int_\Omega |\overline{A}|^2\cdot \mathrm{d}x)^{1/2}(\int_\Omega |\nabla T_\lambda\ \overline{C}|^2\mathrm{d}x)^{1/2} \leqslant \frac{1}{M_0}\int_\Omega |\ \overline{A}|^2\mathrm{d}x + \frac{M_0M_1^2}{4}\int_\Omega |\nabla T_\lambda\overline{C}|^2\mathrm{d}x = \frac{1}{M_0}\int_\Omega |\overline{A}|^2\mathrm{d}x - \frac{M_0M_1^2}{4}\int_\Omega T_\lambda\cdot \overline{C}\Delta T_\lambda\overline{C}\mathrm{d}x = \frac{1}{M_0}\int_\Omega |\overline{A}|^2\mathrm{d}x - \frac{M_0M_1^2}{4}\int_\Omega T_\lambda\overline{C}(\lambda T_\lambda\overline{C} - \overline{C})\mathrm{d}x = \frac{1}{M_0}\int_\Omega |\ \overline{A}|^2\mathrm{d}x - \frac{M_0M_1^2}{4}\lambda\int_\Omega |T_\lambda\ \overline{C}|^2\mathrm{d}x + \frac{M_0M_1^2}{4}\int_\Omega T_\lambda\overline{C}\cdot\overline{C}\mathrm{d}x \leqslant \frac{1}{M_0}\int_\Omega |\overline{A}|^2\mathrm{d}x + \frac{M_0M_1^2}{4}g_\lambda(t).$$

Therefore, from the formula (2. 5), we have

$$g_\lambda(t) = 2\iint_{Q_t}(\lambda T_\lambda\overline{A} - \overline{A} + T_\lambda(\mathrm{div}\,\vec{H}))\overline{C}\mathrm{d}x\mathrm{d}s \leqslant 2M_0\lambda\iint_{Q_t}|\overline{C}T_\lambda\overline{C}|\mathrm{d}x\mathrm{d}s + \frac{M_0M_1^2}{2}\int_0^t g_\lambda(s)\mathrm{d}s.$$

On the other hand, for the solution $u = T_\lambda f$ of the formula (2. 2), it is easily seen that

$$\int_\Omega |\nabla T_\lambda f|^2\mathrm{d}x + \lambda\int_\Omega |T_\lambda f|^2\mathrm{d}x \leqslant C\int_\Omega |f|^2\mathrm{d}x.$$

Using Sobolev's inequality we get

$$\int_\Omega |T_\lambda f|^2\mathrm{d}x \leqslant C\int_\Omega |f|^2\mathrm{d}x.$$

Hence,

$$g_\lambda(t) \leqslant 2M_0\lambda\iint_{Q_t}|\overline{C}T_\lambda\overline{C}|\mathrm{d}x\mathrm{d}s + \frac{M_0M_1^2}{2}\int_0^t g_\lambda(s)\mathrm{d}s \leqslant M_0\lambda\iint_{Q_t}(|\overline{C}|^2 + |T_\lambda\overline{C}|^2)\mathrm{d}x\mathrm{d}s + \frac{M_0M_1^2}{2}\int_0^t g_\lambda(s)\mathrm{d}s \leqslant C_1\lambda\iint_{Q_t}|\overline{C}|^2\mathrm{d}x\mathrm{d}s + \frac{M_0M_1^2}{2}\int_0^t g_\lambda(s)\mathrm{d}s.$$

The application of the Gronwall's inequality yields

$$g_\lambda(t) \leqslant C\lambda,$$

where C is a constant depending only on M_0, M_1, C_1, T and the measure of Ω, but independent on λ and t. Therefore,

$$\lim_{\lambda\to 0} g_\lambda(t) = 0, a.e.\ t \in (0,T),$$

which implies that

$$\overline{C} = C(u_1) - C(u_2) = 0 \text{ a. e. in } Q_T,$$

that means

$$C(u_1(x,t)) = C(u_2(x,t))\ a.e.\ (x,t) \in Q_T.$$

The proof is completed.

From the proof of the theorem 2. 1, we have the following result.

Theorem 2. 2 If both $\Phi_1(s)$ and $\vec{\Phi}_2(s)$ are globally Lipschite continuous, and the set $E = \{s; A'(s) = 0\}$ does not contain any interior point, then the initial boundary value problems (1. 1)~(1. 3) has at most one generalized solution.

Proof Let $u_1, u_2 \in L^\infty(Q_T)$ be two generalized solutions of the problems (1. 1) ~ (1. 3), from Theorems 2. 1 and formula (2. 1), we have

$$\iint_{Q_T}\overline{A}\Delta\varphi\,\mathrm{d}x\mathrm{d}t = \iint_{Q_T}\vec{H}\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t \leqslant (\iint_{Q_T}|\vec{H}|^2\mathrm{d}x\mathrm{d}t)^{1/2}(\iint_{Q_T}|\nabla\varphi|^2\mathrm{d}x\mathrm{d}t)^{1/2} \leqslant M_0M_1(\iint_{Q_T}|\overline{C}|^2\mathrm{d}x\mathrm{d}t)^{1/2}(\iint_{Q_T}|\nabla\varphi|^2\mathrm{d}x\mathrm{d}t)^{1/2} = 0,$$

which implies that

$$\overline{A} = A(u_1) - A(u_2) = 0 \text{ a. e. in } Q_T$$

for the arbitrariness of φ.

Since the set $E = \{s; A'(s) = 0\}$ does not contain any interior point, then $A(s)$ is a strictly increasing function, hence we get

$$u_1(x,t) = u_2(x,t), \text{a. e. } (x,t) \in Q_T.$$

The proof is completed.

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