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New Inequalities on Vertex Folkman Numbers*
关于顶点 Folkman 数的新不等式

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Abstract: For an undirected, simple graph G, and positive integers $a_1,\cdots,a_k$, we write $G\to(a_1,\cdots,a_k)^v$ if and only if for every vertex k-coloring of G, there exists a monochromatic K_{a_i}, for some color $i\in\{1,\cdots,k\}$. The vertex Folkman number is defined as $F_v(a_1,\cdots,a_k;p)=\min\{|V(G)|:G\to(a_1,\cdots,a_k)^v, K_p\not\subset G\}$ for $p>\max\{a_1,\cdots,a_k\}$. In this paper, new recurrent inequalities on vertex Folkman numbers $F_v(k,k;k+1)$ are proved. We also generalize an inequality of Kolev and Nenov on multicolor Folkman numbers.

Key words: vertex Folkman number, upper bound, coloring

摘要:对于无向简单图 G 及正整数 $a_1,\cdots,a_k$,记 $G\to(a_1,\cdots,a_k)^v$ 当且仅当对于图 G 的任意一种顶点 k 染色,一定对某个 $i\in\{1,\cdots,k\}$ 存在顶点全染着颜色 i 的完全子图 K_{a_i}. 对于 $p>\max\{a_1,\cdots,a_k\}$,定义 $F_v(a_1,\cdots,a_k;p)=\min\{|V(G)|:G\to(a_1,\cdots,a_k)^v,K_p\not\subset G\}$ 为顶点 Folkman 数. 证明关于顶点 Folkman 数 $F_v(k,k;k+1)$ 的新的迭代不等式,并推广 Kolev 和 Nenov 的一个关于多色顶点 Folkman 数的不等式.

关键词:顶点 Folkman 数 上界 染色
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1 Introduction

For an undirected, simple graph G, and positive integers $a_1,\cdots,a_k$, we write $G\to(a_1,\cdots,a_k)^v$ ($G\to(a_1,\cdots,a_k)^e$) if and only if for every vertex (edge) k-coloring of G, there exists a monochromatic K_{a_i}, for some color $i\in\{1,\cdots,k\}$.

For positive integers $a_1,\cdots,a_k$ and $p>\max\{a_1,\cdots,a_k\}$, let

$$F_v(a_1,\cdots,a_k;p)=\{G:G\to(a_1,\cdots,a_k)^v,K_p\not\subset G\},$$

$$F_e(a_1,\cdots,a_k;p)=\{G:G\to(a_1,\cdots,a_k)^e,K_p\not\subset G\}.$$

The graphs in $F_v(a_1,\cdots,a_k;p)$ are called vertex Folkman graphs, and the graphs in $F_e(a_1,\cdots,a_k;p)$ are called edge Folkman graphs.

In 1970 Folkman[1] showed that for all r,l and $p>\max\{r,l\}$ the families $F_v(r,l;p)$ and $F_e(r,l;p)$ are nonempty. Folkman's method worked only for two colors. Folkman's theorem was generalized to multicolor case in reference [2] (also see reference [3]).

One can ask what is the minimum number of vertices in a vertex or edge Folkman graph, which leads to the notion of Folkman numbers.

For positive integers $a_1,\cdots,a_k$ and $p>\max\{a_1,\cdots,a_k\}$, the vertex (edge) Folkman number is defined as

$$F_v(a_1,\cdots,a_k;p)=\min\{|V(G)|:G\to(a_1,\cdots,a_k)^v,K_p\not\subset G\};$$

$$F_e(a_1,\cdots,a_k;p)=\min\{|V(G)|:G\to(a_1,\cdots,a_k)^e,K_p\not\subset G\}.$$

Among all vertex Folkman numbers, $F_v(k,k;k+$

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1) and $F_v(3,k;k+1)$ seem more interesting for many researchers. In this paper we will discuss the upper bounds for them.

In reference [4] it was proved that

Theorem 1 For p no less than 3, we have

$F_v(k,k;k+1) \leqslant [k!e]-2$.

In reference [5], the following corollary was proved.

Corollary 1 $F_v(k,k;k+1) \leqslant [2k!(e-1)]-1$.

Theorem 1 in reference [4] was proved earlier than corollary 1. Reference [4] was in Russian and was not well known.

In reference [6], the following theorem and corollary were given.

Theorem 2 For all integer p no less than 2, we have

$F_v(p+1,p+1;p+2) \leqslant (p+1)F_v(p,p;p+1)$.

Corollary 2 For integer p no less than 4, we have

$F_v(p,p;p+1) \leqslant 1.46p!$.

In section 2 we will improve theorem 2 above, based on what we can get new upper bounds better than those in corollary 2.

The following theorem was proved in reference [7].

Theorem 3 If $p \geqslant 3$, then $2p+4 \leqslant F_v(3,p;p+1) \leqslant 4p+2$.

New upper bounds $F_v(3,5;6) \leqslant 22$, $F_v(3,6;7) \leqslant 26$ and $F_v(3,7;8) \leqslant 30$ were gotten in reference [7]. They were used in corollary 3 on $F_v(3,p;p+1)$.

The following theorem was proved in reference [8].

Theorem 4 Let $a_1 \leqslant \cdots \leqslant a_r, r \geqslant 2$ be positive integers and $a_r = b_1 + \cdots + b_s$, where b_i are positive integers too and $b_i \geqslant a_{r-1}, i=1,\cdots,s$. Then

$F_v(a_1,\cdots,a_r;a_r+1) \leqslant \sum_{i=1}^{s} F_v(a_1,\cdots,a_{r-1},b_i;b_i+1)$.

Base on this theorem, the following result was proved in reference [8].

Let $p \geqslant 4$ and $p=4k+l, 0 \leqslant l \leqslant 3$, from the theorem it is easily get that

$F_v(3,p;p+1) \leqslant (k-1)F_v(3,4;5)+F_v(3,4+l;5+l)$.

Then they proved the following corollary (see reference [8]).

Corollary 3 If $p \geqslant 4$ then

$F_v(3,p;p+1) \leqslant \frac{13}{4}p$ for $p \equiv 0 \pmod 4$;

$F_v(3,p;p+1) \leqslant \frac{13p+23}{4}$ for $p \equiv 1 \pmod 4$;

$F_v(3,p;p+1) \leqslant \frac{13p+26}{4}$ for $p \equiv 2 \pmod 4$;

$F_v(3,p;p+1) \leqslant \frac{13p+29}{4}$ for $p \equiv 3 \pmod 4$.

In section 3 we will generalize theorem 4 above. A part of corollary 3 will be improved a little.

2 New upper bound for $F_v(k,k;k+1)$

In this section, we will improve the theorem above.

Theorem 5 For integer k no less than 2, we have

$F_v(2k,2k;2k+1) \leqslant kF_v(2k-1,2k-1;2k)+3k+1$.

Proof Suppose $H \in \mathcal{F}_v(2k-1,2k-1;2k)$, $v(H)=F_v(2k-1,2k-1;2k)$, $V(H)=\{v_1,\cdots,v_n\}$. From $v(H)=n=F_v(2k-1,2k-1;2k)$, we know there is $A \subset V(H)\backslash\{v_1\}$ such that both the subgraph of H induced by A and the subgraph of H induced by $B = V(H)\backslash(\{v_1\} \cup A)$ are K_{2k-1}-free. We might suppose $V(A)=\{v_2,\cdots,v_{n_1}\}$ and $V(B)=\{v_{n_1+1},\cdots,v_n\}$ as well.

Next we will construct a graph G of order $kF_v(2k-1,2k-1;2k)+3k+1$.

Let $V(G)=\bigcup_{i=1}^{3} V(G_i)$ and $V(G_1)=\{w_0\}$.

Let $V(G_2)=\{w_j \mid 1 \leqslant j \leqslant 2k\}$ and $E(G_2)=\{(w_i,w_j) \mid 1 \leqslant i < j \leqslant 2k\}$.

For G_3, $V(G_3)=\bigcup_{i=1}^{k} V(A_i) \cup \bigcup_{i=1}^{k} V(B_i) \cup \{u_1(i) \mid 1 \leqslant i \leqslant k\} \cup \{u_1'(i) \mid 1 \leqslant i \leqslant k\}$, in which for any i satisfies $1 \leqslant i \leqslant k$, $V(A_i)=\{u_2(i),\cdots,u_{n_1}(i)\}$ and $V(B_i)=\{u_{n_1+1}(i),\cdots,u_n(i)\}$.

$E(G_3)=\{(u_y(i),u_z(i)) \mid v_yv_z \in E(H), 1 \leqslant i \leqslant k, 1 \leqslant y < z \leqslant n\} \cup \bigcup_{i=1}^{5} E_i$,

where

$E_1=\{(u_1'(i),u_z(i)) \mid v_1v_z \in E(H), 1 \leqslant i \leqslant k, n_1+1 \leqslant z \leqslant n\}$,

$E_2=\{(u_1'(i),u_z(i+1)) \mid v_1v_z \in E(H), 1 \leqslant i \leqslant k-1, 2 \leqslant z \leqslant n_1\}$,

$E_3=\{(u_1'(k),u_z(1)) \mid v_1v_z \in E(H), 2 \leqslant z \leqslant n_1\}$,

$E_4=\{(u_y(i),u_z(i+1)) \mid v_yv_z \in E(H), 1 \leqslant i \leqslant$

$k-1, n_1+1 \leqslant y \leqslant n, 2 \leqslant z \leqslant n_1\}$,

$E_5 = \{(u_y(k), u_z(1)) \mid v_y v_z \in E(H), n_1+1 \leqslant y \leqslant n, 2 \leqslant z \leqslant n_1\}$.

The set of edges of the graph G is defined by

$$E(G) = \bigcup_{i=2}^{3} E(G_i) \cup E(G_1, G_3) \cup E(G_2, G_3),$$

where

$E(G_1, G_3) = \{(w_0, x) \mid x \in V(G_3)\}$ and $E(G_2, G_3) = \{(u_y(i), w_i) \mid 1 \leqslant y \leqslant n, 1 \leqslant i \leqslant k\} \cup \bigcup_{i=6}^{7} E_i$,

in which

$E_6 = \{(u, w_{i+k}) \mid 1 \leqslant i \leqslant k-1, u \in B_i \cup A_{i+1} \cup \{u_1'(i)\}\}$,

$E_7 = \{(u, w_{2k}) \mid u \in B_k \cup A_1 \cup \{u_1'(k)\}\}$.

From $V(G_3) = \bigcup_{i=1}^{k} V(A_i) \cup \bigcup_{i=1}^{k} V(B_i) \cup \{u_1(i) \mid 1 \leqslant i \leqslant k\} \cup \{u_1'(i) \mid 1 \leqslant i \leqslant k\}$ we know that

$$|V(G_3)| = k(|A_1| + |B_1|) + k + k.$$

From $|A_1| + |B_1| = n-1$, we have $|V(G_3)| = k(|A_1| + |B_1|) + k + k = k(n-1) + 2k = kn + k$. So

$$|V(G)| = \sum_{i=1}^{3} |V(G_i)| = 1 + 2k + kn + k = kn + 3k + 1.$$

From $n = F_v(2k-1, 2k-1; 2k)$ we have

$$|V(G)| = kn + 3k + 1 = kF_v(2k-1, 2k-1; 2k) + 3k + 1.$$

From the construction of graph G, we can see that the subgraph induced by G_3 is K_{2k}-free, and any three vertices in G_2 have no common neighbors in G_3. If two vertices in G_2 have common neighbors in G_3, then the subgraph of G induced by their common neighbors in G_3 must be K_{2k-1}-free, because it is isomorphic to the subgraph of H induced by A or B.

From all these we can see G is K_{2k+1}-free.

Now, we give G any red-blue vertex coloring, i.e., we color any vertex in G with red or blue. We might suppose w_0 is in color red as well.

We will prove $G \to (2k, 2k)^v$. We know the subgraph of G induced by G_2 is a complete subgraph on $2k$ vertices. If there is not monochromatic complete subgraph on $2k$ vertices in G, there is at least one vertex w_i in G_2 is in color blue.

From the construction of graph G we know that the subgraph of G induced by the common neighbors of w_0 and w_i are isomorphic to H. So there is not monochromatic complete subgraph on $2k-1$ vertices in H, what contradicts with $H \in \boldsymbol{F}_v(2k-1, 2k-1; 2k)$. Therefore we have $G \to (2k, 2k)^v$.

Because G is K_{2k+1}-free, $G \to (2k, 2k)^v$ and $|V(G)| = kF_v(2k-1, 2k-1; 2k) + 3k + 1$, we have

$$F_v(2k, 2k; 2k+1) \leqslant kF_v(2k-1, 2k-1; 2k) + 3k + 1.$$

It is natural to get the following theorem with similar method.

Theorem 6 Suppose k is an integer no less than 2, $H \in F_v(2k, 2k; 2k+1)$, $\upsilon(H) = F_v(2k, 2k; 2k+1)$. Let $\{v_1\} \subset H$, $A \subset V(H) \setminus \{v_1\}$, G_1 be the subgraph of H induced by A and G_2 be the subgraph of H induced by $V(H) \setminus (\{v_1\} \cup A)$. Suppose both G_1 and G_2 are K_{2k}-free. If x is the order of the maximum isomorphic induced subgraphs of G_1 and G_2, then we have

$$F_v(2k+1, 2k+1; 2k+2) \leqslant (k+1)F_v(2k, 2k; 2k+1) - x + 3k + 2.$$

In reference [9] the following theorem and corollary were proved.

Theorem 7 Let $a_1, \cdots, a_k, b_1, \cdots, b_k, p, q$ be positive integers, $\max\{a_1, \cdots, a_k\} \leqslant p$, $\max\{b_1, \cdots, b_k\} \leqslant q$, then

$$F_v(a_1 b_1, \cdots, a_k b_k; pq+1) \leqslant F_v(a_1, \cdots, a_k; p+1) \cdot F_v(b_1, \cdots, b_k; q+1).$$

Corollary 4 Let a, b be positive integers no less than 2, $a \leqslant p, b \leqslant q$, then

$$F_v(ab, ab; pq+1) \leqslant F_v(a, a; p+1) F_v(b, b; q+1).$$

Although corollary 4 is a powerful tool to give upper bound for $F_v(k, k; k+1)$, sometimes it is better to use it together with theorem 5 or 6. For instance, let $p = ab + 1$, and we want to get an upper bound for $F_v(p, p; p+1)$. If we have good upper bounds for $F_v(a, a; a+1)$ and $F_v(b, b; b+1)$ respectively, then we can get an upper bound for $F_v(ab, ab; ab+1)$ by corollary 4 firstly, and give $F_v(p, p; p+1)$ an upper bound by theorem 5 or 6.

3 A generalization of theorem 4

We know $F_v(3, 3; 4) = 14$ (see reference [10]) and $F_v(3, 4; 5) = 13$ (see reference [11]).

In reference [8], $F_v(3, p; p+1) \leqslant \frac{13p + 29}{4}$ for $p \equiv 3 \pmod 4$ in corollary 3 was proved using $F_v(3, 7; 8) \leqslant 4 \times 7 + 2 = 30$.

But use theorem 4 in the same paper [7] we have

$$F_v(3, 7; 8) \leqslant F_v(3, 3; 4) + F_v(3, 4; 5) \leqslant 14 + 13 = 27.$$

From $F_v(3, 7; 8) \leqslant 27$ and their inequality (see reference [8])

$$F_v(3,p;p+1) \leqslant (k-1)F_v(3,4;5) + F_v(3,4+l;5+l),$$

we have

Corollary 5 If $p \geqslant 4$ then

$$F_v(3,p;p+1) \leqslant \frac{13p+17}{4} \text{ for } p \equiv 3 \pmod 4.$$

Now we will give theorem 4 a natural generalization, which can be proved by the same method.

Theorem 8 Let $a_1 \leqslant \cdots \leqslant a_r, r \geqslant 1$ be positive integers and $p_i \geqslant \max\{a_r, b_i\}, i=1,\cdots,s$, where p_i and b_i are positive integers too. Then

$$F_v(a_1,\cdots,a_r, \sum_{i=1}^{l} b_i; \sum_{i=1}^{l}(p_i-1)+1) \leqslant \sum_{i=1}^{l} F_v(a_1,\cdots,a_r,b_i;p_i).$$

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科学家发现汞污染新机制

汞,这种能对神经产生毒害的元素往往存在于自然界的土壤和植被当中,当发生森林火灾时,许多汞会随烟四散,而且常常会飘移几十万公里。然而,也有一部分汞会"留守"在火灾地点周围的生态环境中,它们被冲入附近的湖泊,最终在鱼类体内沉积下来。据加拿大埃德蒙顿阿尔伯塔大学的一项最新研究显示,森林大火可以通过一种令人惊讶的机制——使水生生态系统出现紊乱,从而导致鱼体内的汞浓度升高。

加拿大埃德蒙顿阿尔伯塔大学的研究人员对 1120hm² 区域内暴发森林大火产生的汞对生态系统产生的影响进行研究。在森林大火暴发后为期1个月的汞扩散过程中,短时间内有大量的汞以一种类似"脉冲"的方式通过被火灾侵蚀的土壤进入到在贾斯珀国家公园"摩押湖"中,同时,地表径流也使湖中氮的总量翻了一番,而磷的浓度则变为原来的4倍。湖中的无脊椎动物通过这些营养物质大量繁殖,导致食物链中所有环节的成员数量都大幅增加,尤其是小鳟鱼,更是"人丁兴旺"。这些因素导致了新的捕食行为,从而改变了食物链。湖中充足的幼虹鳟基本被4种靠无脊椎动物生存的鱼捕食殆尽,因此,那些鱼的体内积累了比平常更多的汞。以成年虹鳟为例,它们体内的汞含量比火灾发生前高出5倍之多。处于食物链顶端的捕食者——湖鳟,也在大火之后开始吃一种名为加拿大白鳟的鱼,这使得它体内的汞含量更高。这些鳟鱼体内的汞含量超过了美国环保局认定的安全数字,然而,附近处于火灾区域之外的湖泊,其食物链和汞的水平都没有发生变化。

生态学家认为,这项研究说明森林大火对食物链有双重影响,即汞扩散之后也增加了其在食物链中的积累,而通过食物链流动的汞可能比我们想象的更多。

(据《科学时报》)