Almost Periodic Solutions for some Nonlinear Integrodifferential Equations*

一类非线性积分微分方程的概周期解

LIU Yong-jian[1], FENG Chun-hua[2]

刘永建[1]，冯春华[2]

(1. Department of Mathematics and Computer Science, Yulin Normal College, Yulin, Guangxi, 537000, China; 2. Department of Mathematics, Guangxi Normal University, Guilin, Guangxi, 541004, China)

(1. 玉林师范学院数学与计算机科学系，广西玉林 537000；2. 广西师范大学数学科学学院，广西桂林 541004)

Abstract: This paper investigated a nonlinear integrodifferential equation. The existence of almost periodic solutions for the equation is obtained.

Key words: integrodifferential equation, almost periodic solution, existence-uniqueness

摘要：讨论一类非线性积分微分方程的概周期解，结合运用李雅普诺夫泛函、不动点原理和概周期函数的有界性，给出系统存在唯一概周期解的一组充分条件.

关键词：积分微分方程　概周期解　存在唯一性

中图法分类号：O175　　**文献标识码**：A　　**文章编号**：1005-9164(2006)04-0264-03

1 Introduction

Existence of periodic and almost periodic solutions for nonlinear differential equations have been investigated by many authors[1～6]. For the qualification of almost periodic differential equations, one can see in reference[7～9]. In this paper, we shall extend the results of reference[1, 2] to the following nonlinear integrodifferential equations. Since periodic system is a special case of almost periodic system, we also extend the results of reference[4, 5]. Consider the following system

$$x'(t) = A(t,x(t))x(t) + \int_{-\infty}^{t} C(t,s)x(s)\mathrm{d}s + g(t,x(t)) + p(t), \tag{1}$$

where $-\infty < s \leqslant t, t \in R, x \in R^n$. Specially, if $A(t, x(t)) = A(t)$ and $g(t,x) = g(t)$, Burton[4] and Huang[5] have investigated the existence of periodic solutions. Under the condition that the system (1) has a unique bounded, we shall obtain some result on the existence of a unique almost periodic solution of system (1).

Throughout this paper, we assume that $A(t, x(t))$ is a $n \times n$ continuous matrix almost periodic in t uniformly for x, $p(t)$ is a vector continuous almost periodic function, vector $g(t,x)$ is almost periodic in t uniformly for x, $C(t,s)$ is a $n \times n$ continuous almost periodic matrix, namely, for any $\varepsilon > 0$ and any compact set K in $R \times R$, there exists an $L = L(\varepsilon, K) > 0$ such that any interval of length L contains an τ for which

$$|C(t+\tau, s+\tau) - C(t,s)| \leqslant \varepsilon, t, s \in R. \tag{2}$$

If $x = (x_1, x_2, \cdots, x_n)^T \in R^n$, $A = (a_{ij})_{n\times n}$, we define

$$\|x\| = \sum_{i=1}^{n} |x_i|, \quad \|A\| = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|.$$

2 Main results

For system (1), we impose the following assumptions:

(i) For any x, there exists constants $\delta > 0$ such that

$$a_{jj}(t,x) + \sum_{i=1, i\neq j}^{n} |a_{ij}(t,x)| + \sum_{i=1}^{n} \int_{t}^{+\infty} |C_{ij}(t,s)|\mathrm{d}s \leqslant -\delta, j = 1, 2, \cdots, n. \tag{3}$$

收稿日期：2006-08-01

修回日期：2006-10-26

作者简介：刘永建(1975-)，男，湖南新邵人，硕士，主要从事微分方程周期解和概周期解的研究工作.

* Supported by NSF of China (10461003).

(i*)

$$-a_{jj}(t,x)+\sum_{i=1,i\neq j}^{n}|a_{ij}(t,x)|+\sum_{i=1}^{n}\int_{t}^{+\infty}|C_{ij}(t,s)|ds\leqslant-\delta,j=1,2,\cdots,n. \quad (4)$$

(ii) There exists a constant $k(0<k<\delta)$ such that for any $x,y\in R^n$, we have

$$\|g(t,x)-g(t,y)\|\leqslant k\|x-y\|. \quad (5)$$

(iii) There exists $L>0$ such that $\|g(t,0)+p(t)\|\leqslant L$.

We shall denote the function space consisting of all translates of f by $T(f)$, that is, $f_\tau\in T(f)$, where

$$f_\tau(t,x)=f(t+\tau,x),\tau\in R.$$

Lemma 1[9] Let $f(t,x)\in C(R\times D,R^n)$ be almost periodic in t uniformly for $x\in D$, where D is a subset in R^n. Then, for any real sequence h'_k, there exists a subsequence h_k of h'_k and a continuous function $g(t,x)$ such that

$$f(t+h_k,x)\to g(t,x), \quad (6)$$

uniformly on $R\times S$ as $k\to\infty$, where S is any compact set in D. Moreover, $g(t,x)$ is also almost periodic in t uniformly for $x\in D$.

Let $H(f)$ denote the uniform closure of $T(f)$ in the sense of Formula (6), namely, $g\in H(f)$ means $\lim_{k\to\infty}f(t+h_k,x)=g(t,x)$ for some sequence h_k.

Lemma 2[1] If for any $\tilde{A}(t)\in H(A),\tilde{C}(t,s)\in H(C)$ and $\tilde{p}(t)\in H(p)$, the equation

$$x'=\tilde{A}(t)x+\int_{-\infty}^{t}\tilde{C}(t,s)x(s)ds+\tilde{p}(t) \quad (7)$$

has a unique bounded solution on R in a compact set K, then there is a almost periodic solution of the following equation

$$x'=A(t)x+\int_{-\infty}^{t}C(t,s)x(s)ds+p(t). \quad (8)$$

Theorem 1 Suppose that the conditions (i), (ii) and (iii) are satisfied, then there exists a unique bounded solution of system (1).

Proof Let $x(t)$ be a solution of system (1), consider a Liapunov function

$$V(t)=\|x(t)\|=\sum_{i=1}^{n}|x_i(t)|, \quad (9)$$

then calculate the right derivative of $V(t)$ we have

$$D^+V(t)=\sum_{i=1}^{n}\operatorname{sgn}(x_i(t))\cdot x'_i(t)=$$

$$\sum_{i=1}^{n}\operatorname{sgn}(x_i(t))[\sum_{j=1}^{n}a_{ij}(t,x)x_j(t)+\int_{-\infty}^{t}\sum_{j=1}^{n}C_{ij}(t,s)x_j(s)ds+g_i(t,x)+p_i(t)]\leqslant a_{11}(t,x)|x_1(t)|+|a_{12}(t,x)||x_2(t)|+\cdots|a_{1n}(t,x)||x_n(t)|+\int_{-\infty}^{t}|C_{11}(t,s)|ds|x_1(t)|+\int_{-\infty}^{t}|C_{12}(t,s)|ds|x_2(t)|+\cdots+\int_{-\infty}^{t}|C_{1n}(t,s)|ds|x_n(t)|+|g_1(t,x)|+|p_1(t)|+\cdots+a_{n1}(t,x)|x_1(t)|+|a_{n2}(t,x)||x_2(t)|+\cdots|a_{nn}(t,x)||x_n(t)|+\int_{-\infty}^{t}|C_{n1}(t,s)|ds|x_1(t)|+\int_{-\infty}^{t}|C_{n2}(t,s)|ds|x_2(t)|+\cdots+\int_{-\infty}^{t}|C_{nn}(t,s)|ds|x_n(t)|+|g_n(t,x)|+|p_n(t)|\leqslant\sum_{j=1}^{n}[a_{jj}(t,x)+\sum_{i=1,j\neq i}^{n}|a_{ij}(t,x)|+\sum_{i=1}^{n}\int_{-\infty}^{t}|C_{ij}(t,s)|ds]|x_j(t)|+\sum_{i=1}^{n}|g_i(t,x)-g_i(t,0)|+\sum_{i=1}^{n}(|g_i(t,0)|+|p_i(t)|)$$

$$\leqslant-\delta\sum_{j=1}^{n}|x_j(t)|+k\|x(t)\|+L=-(\delta-k)V(t)+L. \quad (10)$$

Since $k<\delta$, therefore, $-(\delta-k)=-c<0$. So we have

$$D^+V(t)\leqslant-cV(t)+L. \quad (11)$$

We clain that

$$V(t)\leqslant L/c \text{ for } t\geqslant 0. \quad (12)$$

Otherwise, there exists $t_1>0$ such that

$$V(t)\leqslant L/c(0\leqslant t<t_1),V(t_1)>L/c. \quad (13)$$

From Formula (11), we have

$$D^+V(t)\leqslant 0 \text{ for } 0\leqslant t<t_1. \quad (14)$$

Hence for $0\leqslant t<t_1$, we get

$$V(t)\leqslant V(0)\leqslant L/c,0\leqslant t<t_1. \quad (15)$$

Let $t\to t_1$, we have $V(t_1)\leqslant L/c$. This contradicts with Formula (13). Since $V(t)=\|x(t)\|$, this means $x(t)(t\in R^+)$ is bounded. Now consider sequence $\{t_n\},t_n\to+\infty$ as $n\to\infty$, then $\{x(t+t_n)\}$ is uniformly bounded on R^+, and

$$|x'(t+t_n)|\leqslant\sup_{t\geqslant0}|A(t,x(t))|\cdot|x(t)|+\sup_{t\geqslant0}\int_{-\infty}^{t}|C(t,s)|ds\cdot\sup_{t\geqslant0}|x(t)|+\sup_{t\geqslant0}|g(t,x(t))|+L, \quad (16)$$

which implies that $\{x(t+t_n\}$ is uniformly bounded and equicontinuous. It follows from the Ascoli theorem that, $\{x(t+t_n\}$ locally uniformly converges to $\bar{x}$ on R. Moreover,

$$x'(t+t_n)=A(t+t_n,x(t+t_n))x(t+t_n)+\int_{-t_n}^{t}C(t+t_n,s+t_n)x(s+t_n)ds+g(t+t_n,x(t+t_n))+p(t+t_n). \quad (17)$$

Let $n\to\infty$, we have

$$\bar{x}'(t)=A(t,\bar{x}(t))\bar{x}(t)+\int_{-\infty}^{t}C(t,s)\bar{x}(s)ds+g(t,\bar{x}(t))+p(t), \quad (18)$$

that is, $\bar{x}(t)$ is a solution of system (1). Since $\{x(t+t_n\}$ is uniformly bounded, thus $\bar{x}(t)$ is bounded for $t\geqslant-t_n$. Let $n\to\infty$ we have $\bar{x}(t)$ is bounded for $t\in R$. Namely, there is a positive constant $M(=L/c)$ such that $\|\bar{x}(t)\|\leqslant M$ for $t\in R$.

Now suppose that $z(t)$ is a solution of system(1), consider a Liapunov function

$$W(t) = \| z(t) - \bar{x}(t) \| = \sum_{i=1}^{n} |z_i(t) - \bar{x}_i(t)|. \quad (19)$$

Then from the right derivative of $W(t)$ we easily get

$$D^+ W(t) = \sum_{i=1}^{n} \operatorname{sgn}(z_i(t) - \bar{x}_i(t)) \cdot (z'_i(t) - \bar{x}'_i(t)) \leqslant - cW(t), \quad (20)$$

which implies that $z(t) = \bar{x}(t)$ for any $t \in R$. Thus, there is a unique bounded solution in system(1).

Theorem 2 Under the assumptions (i), (ii) and (iii), there exists a unique almost periodic solution of system(1).

Proof Let $B = \{u(t) | u(t)$ is almost periodic$\}$. Define the norm $\| u(t) \| = \sup_{t \in R} |u(t)|$, then B is a Banach space. Note that Lamma 2 and Theorem 1 still hold under the assumptions (i), (ii) and (iii). Thus, for any $u(t) \in B, \tilde{A}(t) \in H(A), \tilde{C}(t,s) \in H(C)$, $\tilde{g}(t) \in H(A)$, and $\tilde{p}(t) \in H(p)$, from Lemma 2 and Theorem 1, we know that

$$x'(t) = A(t,u(t))x(t) + \int_{-\infty}^{t} C(t,s)x(s)\mathrm{d}s + g(t,u(t)) + p(t), \quad (21)$$

system (21) admits a unique almost periodic solution denoted by $x_u(t)$. So we can define operator $T: B \rightarrow B$ as follows:

$$T: u(t) \rightarrow x_u(t).$$

Let $B_n = \{u(t); u(t) \in B, \| u \| \leqslant n\}$, where n is a natural number. It is easily from Formula(11) to know that there exists a natural number N sufficiently large, for example, $N > \frac{L}{c} + 1$ such that $T: B_N \rightarrow B_N$.

Now we show that T is a compact operator. For any sequence $u_n(t)$ satisfying $\| u_n(t) \| \leqslant N (n = 1, 2, \cdots)$, since $Tu_n(t)$ is the solution of system

$$x'(t) = A(t,u_n(t))x(t) + \int_{-\infty}^{t} C(t,s)x(s)\mathrm{d}s + g(t,u_n(t)) + p(t). \quad (22)$$

We have

$$|(Tu_n)'(t)| \leqslant \sup_{\|x\| \leqslant N} \| A(t,u_n(t)) \| \cdot \| x(t) \| + \sup_{\|x\| \leqslant N} \int_{-\infty}^{t} |C(t,s)| \mathrm{d}s \cdot \sup_{\|x\| \leqslant N} \| x \| + \sup_{\|x\| \leqslant N} \| g(t, x(t)) \| + L, \quad (23)$$

which implies that $\{Tu_n(t)\}$ is uniformly bounded and equicontinuous. Thus, TB_N is a compact subset of B, namely, T is a compact operator. T is a continuous operator. For any $u(t), v(t) \in B_N$, it follows from Formula(10) that

$$\| Tu - Tv \| \leqslant \frac{1}{\delta} \| g(t,u) - g(t,v) \|. \quad (24)$$

Since $g(t,x)$ is almost periodic in t uniformly for x, so $g(t,x)$ is continuous function. This implies that T is a continuous operator. Thus, there exists an almost periodic solution of system (1) from The Schauder fixed point theorem. Since almost periodic solution is bounded and system (1) has a unique bounded solution, this means that there exists a unique almost periodic solution of system (1). The proof is completed.

Remark Similarly, if condition (i) replaced by (i)*, Theorem 2 still holds.

Example Consider the following scalar equation

$$x'(t) = -(4 + \cos t - \cos \pi t)x(t) + \frac{1}{5}\int_{-\infty}^{t} (\cos t + \cos \pi t)\mathrm{e}^{-t+s} x(s)\mathrm{d}s + \frac{x(t)\sin t}{1 + \mathrm{e}^{x^2}(t)} + \sin \pi t. \quad (25)$$

Note that $\cos t - \cos \pi t$ is an almost periodic function. One can verify that the conditions of Theorem 2 are satisfied. So there exists a unique almost periodic solution of equation (25).

References:

[1] ZENG W. The existence of almost periodic solutions of Volterra integrodifferential equations with infinite delay [J]. Ann of Diff Eqs, 1991, 9(2): 240-249.

[2] WANG Q. Existence and uniqueness and instability of almost periodic solutions to nonlinear systems [J]. J Huaqiao University: Natural Sciences, 1997, 18(4): 341-346.

[3] CHEN F. Periodic solutions of nonlinear integrodifferential equations with infinite delay [J]. Acta Mathematicae Applicatae Sinica, 2003, 26(1): 141-148.

[4] BURTON T A. Periodic solutions of integrodifferential equations[J]. J London Math Soc, 1985, 31: 527-548.

[5] HUANG Q. The existence of periodic solution for the functional differential equations with infinite delay[J]. Chinese Science, 1984, 10(A): 882-889.

[6] HAMAYA Y. Total stability property in limiting equations of integrodifferential equations[J]. Funkcialaj Ekvacioj, 1990, 33: 345-362.

[7] ZHENG ZU XIU. Theory of functional differential equations[M]. Hefei: Anhui Educational Press, 1994.

[8] HE CONG YOU. Almost periodic differential equations [M]. Beijing: High Educational Press, 1992.

[9] YOSHIZAWA T. Stability theory and the existence of periodic solutions and almost periodic solutions[M]. New York: Springer-Verlag, 1975.

(责任编辑:邓大玉　凌汉恩)