Almost Periodic Solutions of Linear Impulsive Delay Differential Equations*

一类具有脉冲效应的时滞线性微分方程的概周期解

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Abstract: This paper deals with the existence of almost periodic solutions of linear impulsive delay differential equations as follows,

$$\begin{cases} x' = A(t)x(t) + \sum_{i=1}^{n} A_i(t)x(t-r_i) + h(t), t \neq \tau_k, \tau_k < \tau_{k+1}, k = \pm 1, \pm 2L, \\ \Delta x(t) = B_k x(t) + I_k(x(t)) + p_k, t = \tau_k, \end{cases}$$

by using the fixed point theorem of contraction mapping principle we present a set of sufficient conditions to ensure the existence and uniqueness of almost periodic solutions of impulsive delay differential equations in some closed convex set.

Key words: differential equation, fixed point theorem, almost periodic solution, existence and uniqueness

摘要:基于压缩不动点原理,考虑一类具有脉冲效应的时滞微分方程

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概周期解的存在性问题,在一定条件下获得了系统存在唯一概周期的一组充分条件.

关键词:微分方程 不动点原理 概周期 存在性
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In real world, many problems and phenomena can be described with impulsive delay differential equations. Recently, impulsive delay differential equations have been intensively researched[1~3]. Some qualitative properties such as oscillations, asymptotic behavior and stability are investigated by many authors (for example, see reference [4 ~ 8] and references cited therein).

To the author's knowledge, there are few works on almost periodicity of impulsive delay differential equations. The study for the almost periodicity of the delay differential equations with impulsive effects is still in an initial stage of its development[9~12].

In this paper, we investigated the existence of almost periodic solutions for linear impulsive delay differential equations as follows:

$$\begin{cases} x' = A(t)x(t) + \sum_{i=1}^{n} A_i(t)x(t-r_i) + \\ \qquad h(t), t \neq \tau_k, \tau_k < \tau_{k+1}, k = \pm 1, \pm 2, \cdots \\ \Delta x(t) = B_k x(t) + I_k(x(t)) + p_k, t = \tau_k, \end{cases} \tag{1}$$

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where $A(t), A_i(t) \in C(R, R^{n\times n})(i = 1, 2, \cdots, n)$. $h(t) \in C(R^n)$, $B_k \in R^{n\times n}$, $I_k \in C(R, R^n)$, $p_k \in R^n$.

Let $PC(J, R^n)$, $J \in R$ be the set of all piecewise continuous functions $x: J \to R^n$ with points of discontinuity of first kind τ_k in which it is left continuous, namely,

$x(\tau_k - 0) = x(\tau_k)$, $\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0)$, $k \in Z$.

Together with the system (1) we first consider the linear system

$$\begin{cases} x' = A(t)x(t), t \neq \tau_k, \\ \Delta x(t) = B_k x(t), t = \tau_k. \end{cases} \quad (2)$$

Definition 1[1] Let $B = \{\tau_k\}$, and the set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in Z$, $j \in Z$, $\{\tau_k\} \in B$ is said to be uniformly almost periodic if for arbitrary $\varepsilon > 0$ there exists relatively dense set of ε-almost periodic common for any sequences.

Definition 2[2] The function $\varphi \in PC(R, R^n)$ is said to be almost periodic, if:

(a) The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in Z$, $j \in Z$, $\{\tau_k\} \in B$ is uniformly almost periodic.

(b) For any $\varepsilon > 0$ there exists a real number $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuity of $\varphi(t)$ and satisfy the inequality $|t' - t''| < \delta$, then $|\varphi(t') - \varphi(t'')| < \varepsilon$.

(c) For any $\varepsilon > 0$ there exists a relatively dense set T such that if $\tau \in T$, Then $|\varphi(t + \tau) - \varphi(t)| < \varepsilon$ for all $t \in R$ satisfying the condition $|t - \tau_k| > \varepsilon$, $k \in Z$.

The elements of T are called ε-almost periodic of $\varphi(t)$.

Recall [2] that if $U_k(t, s)$ is the Cauchy matrix for the following system

$$x'(t) = A(t)x(t), \tau_{k-1} < t \leqslant \tau_k, \{\tau_k \in B\}, \quad (3)$$

then the Cauchy matrix for the system (2) is in the form

$$W(t,s) = \begin{cases} U_k(t,s), T_{k-1} < s \leqslant t \leqslant \tau_k, \\ U_{k+1}(t, \tau_k + 0)(E + B_k)U_k(t,s), \\ \qquad \tau_{k-1} < s \leqslant \tau_k < t \leqslant \tau_{k+1}, \\ U_{k+1}(t, \tau_k + 0)(E + B_k)U_k(\tau_k, \\ \qquad \tau_k + 0)\cdots(E + B_i)U_i(\tau_i, s), \\ \qquad \tau_{i-1} < s \leqslant \tau_i < \tau_k < t \leqslant \tau_{k+1}, \end{cases} \quad (4)$$

and the solutions of (2) are written in the form

$$x(t; t_0, x_0) = W(t, t_0)x_0. \quad (5)$$

For system (1), we introduce the following conditions,

H1 $A(t) \in C(R, R^{n\times n})$ and $A_i(t) \in (R, R^{n\times n})(i = 1, 2, \cdots, n)$ are almost periodic.

H2 Det $(E + B_k) \neq 0 (E \in R^{n\times n})$ and the sequence $\{B_k\}$ is almost periodic.

H3 The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in Z$, $j \in Z$, $\{\tau_k\} \in B$ is uniformly almost periodic and there exists $\theta > 0$ such that $\inf_k\{\tau_{k+1} - \tau_k\} = \theta > 0$.

H4 The function $h(t)$ is almost periodic, $\{p_k\}$ is almost periodic sequence and there exists a positive constant C_0 such that $\max\{\sup_{t\in R}|h(t)|, \max_k|p_k|\} < C_0$.

H5 The sequence of functions $I_k(x)$ is almost periodic uniformly with respect to $x \in R$ and there exists $L > 0$ such that $|I_k(x) - I_k(y)| \leqslant L|x - y|$.

Lemma 1[1] Suppose that the conditions H1~H5 hold, then for each $\varepsilon > 0$, there exist $\varepsilon_1 (0 < \varepsilon_1 < \varepsilon)$ and relatively dense sets T of real numbers and Q of whole numbers such that the following relations hold,

(1) $|A(t + \tau) - A(t)| < \varepsilon$, $|A_i(t + \tau) - A_i(t)| < \varepsilon$, $\tau \in T$, $i = 1, 2, \cdots, n$;

(2) $|B_{k+q} - B_k| < \varepsilon$, $t \in R$, $q \in Q$, $k \in Z$;

(3) $|p_{k+q} - p_k| < \varepsilon$, $|\tau_k^q - \tau| < \varepsilon_1$, $q \in Q$, $\tau \in T$, $k \in Z$;

(4) $|h(t + \tau) - h(t)| < \varepsilon$, $t \in R$, $\tau \in T$ and $|t - \tau_k| > \varepsilon$, $k \in Z$.

Lemma 2[1] Suppose that there exist positive constants K and λ such that

$$|W(t,s)| \leqslant Ke^{-\lambda(t-s)}, t \geqslant s, \quad (6)$$

then for any $\varepsilon > 0$, $t \geqslant s$, $|t - \tau_k| > \varepsilon$, $|s - \tau_k| > \varepsilon$, $\tau \in T$, there exists a positive constant Γ such that the following inequality holds:

$$|W(t + \tau, s + \tau) - W(t,s)| \leqslant \varepsilon\Gamma e^{-(\lambda/2)(t-s)}, t \geqslant s. \quad (7)$$

Under the above assume conditions, we have

Theorem 1 Suppose that

(Ⅰ) Conditions H1~H5 and inequalities (6) and (7) hold;

(Ⅱ) Let L_i be denoted the norm of $A_i(t)$, and the number

$$\gamma = K\left(\frac{1}{\lambda}\sum_{i=1}^{n} L_i + \frac{L}{1 - e^{-\lambda}}\right) < 1, \quad (8)$$

then system (1) has a unique almost periodic solution in some closed convex set.

Proof We denote $D, D \subset PC(R, R^n)$ the set of all almost periodic functions $u(t)$ satisfying the inequality $\|u\| < M, \|u\| = \sup_{t\in R}|u(t)|$, where $M = KC_0(\frac{1}{\lambda} + \frac{1}{1-e^{-\lambda}})$.

Define an operator S in D as follows,

$$Su = \int_{-\infty}^{t} W(t,s)[\sum_{i=1}^{n} A_i(s)u(s-r_i) + h(s)]ds + \sum_{\tau_k<t} W(t,\tau_k)[I_k(u(\tau_k)) + p_k] \quad (9)$$

and subset $D^*, D^* \subset D, D^* = \{u \in D, \|u - u_0\| \leqslant \frac{\gamma M}{1-\gamma}\}$, where

$$u_0 = \int_{-\infty}^{t} W(t,s)h(s)ds + \sum_{\tau_k<t} W(t,\tau_k)p_k. \quad (10)$$

We get

$$\|u_0\| \sup_{t\in R}\{\int_{-\infty}^{t}|W(t,s)\|h(s)|ds + \sum_{\tau_k<t}|W(t,\tau_k)\|p_k|\} \leqslant \sup_{t\in R}\{\int_{-\infty}^{t} Ke^{-\lambda(t-s)}|h(s)|ds + \sum_{\tau_k<t} Ke^{-\lambda(t-\tau_k)}|p_k|\} \leqslant K(\frac{C_0}{\lambda} + \frac{C_0}{1-e^{-\lambda}}) = M. \quad (11)$$

Then for any $u \in D^*$, from (10) and (11), we have

$$\|u\| \leqslant \|u - u_0\| + \|u_0\| \leqslant \frac{\gamma M}{1-\gamma} + M = \frac{M}{1-\gamma}. \quad (12)$$

Now we prove that S is a self mapping from D^* to D^*. For arbitrary $u \in D^*$ it follows

$$\|Su - u_0\| = \sup_{t\in R}\{\int_{-\infty}^{t}|W(t,s)|[\sum_{i=1}^{n} A_i(s)u(s-r_i)]ds + \sum_{\tau_k<t}|W(t,\tau_k)||I_k(u(\tau_k))|\} \leqslant$$
$$\sup_{t\in R}\{\int_{-\infty}^{t} Ke^{-\lambda(t-s)}\sum_{i=1}^{n} L_i\|u\|ds + \sum_{\tau_k<t} Ke^{-\lambda(t-\tau_k)}L\|u\|\} \leqslant$$
$$K(\frac{1}{\lambda}\sum_{i=1}^{n} L_i + \frac{L}{1-e^{-\lambda}})\|u\| \leqslant \gamma\|u\| \leqslant \frac{\gamma M}{1-\gamma}. \quad (13)$$

Let $\tau \in T, q \in Q$ where the sets T and Q are determined in Lemma 1. Then

$$\|Su(t+\tau) - Su(t)\| \leqslant \sup_{t\in R}\{\int_{-\infty}^{t}|W(t+\tau, s+\tau) - W(t,s)|(\sum_{i=1}^{n}|A_i(s+\tau)||u(s+\tau-r_i)|)ds + \int_{-\infty}^{t}|W(t,s)|(\sum_{i=1}^{n}|A_i(s+\tau)||u(s+\tau-r_i)|) - u(s-r_i)|)ds + \int_{-\infty}^{t}|W(t,s)|(\sum_{i=1}^{n}|A_i(s+\tau) - A_i(s)||u(s-r_i)|)ds + \sum_{\tau_k<t}|W(t+\tau,\tau_{k+q}) - W(t,\tau_k)||I_{k+q}(u(\tau_{k+q}))| + \sum_{\tau_k<t}|W(t,\tau_k)||I_{k+q}(u(\tau_{k+q})) - I_k(u(\tau_k))| \leqslant$$
$$\varepsilon(\frac{2\Gamma M}{\lambda(1-\gamma)}\sum_{i=1}^{n} L_i + \frac{K}{\lambda}\sum_{i=1}^{n} L_i + \frac{KnM}{\lambda(1-\gamma)} + \frac{L\Gamma N}{1-e^{-\lambda}} + \frac{L}{1-e^{-\lambda}})\}, \quad (14)$$

where N is a natural number. Since ε is sufficiently small, then $\varepsilon(\frac{2\Gamma M}{\lambda(1-\gamma)}\sum_{i=1}^{n} L_i + \frac{K}{\lambda}\sum_{i=1}^{n} L_i + \frac{KnM}{\lambda(1-\gamma)} + \frac{L\Gamma N}{1-e^{-\lambda}} + \frac{L}{1-e^{-\lambda}})$ is also sufficiently small. This means that $Su \in D^*$, namely, S is a mapping from D^* to D^*.

For any $u \in D^*, v \in D^*$, we get

$$\|Su - Sv\| \leqslant \sup_{t\in R}\{\int_{-\infty}^{t}|W(t,s)|(\sum_{i=1}^{n}|A_i(s)||u(s-r_i) - v(s-r_i)|)ds + \sum_{\tau_k<t}|W(t,\tau_k)||I_k(u(\tau_k)) - I_k(v(\tau_k))|\} \leqslant K(\frac{1}{\lambda}\sum_{i=1}^{n} L_i + \frac{L}{1-e^{-\lambda}})\|u - v\| = \gamma\|u - v\|. \quad (15)$$

Since $\gamma < 1$, then S is a contracting operator in D^*. Therefore, there exists a unique almost periodic solution of system (1) in the region D^*. The proof is completed.

Example 1 Consider the following linear delay differential equations

$$\begin{cases} x'_1(t) = -(10 - \sin t)x_1(t) + \cos\pi t x_2(t) - \sin t x_1(t - \frac{1}{10}) - \cos\pi t x_2(t - \frac{1}{10}) + \frac{1}{2}\cos t, \\ x'_2(t) = \sin\pi t x_1(t) - (9 + \cos t)x_2(t) + \cos\pi t x_2(t - \frac{1}{10}) + \sin t, t \neq k\pi, k = \pm 1, \pm 2, \cdots \end{cases} \quad (16)$$

with

$$\begin{cases} x_1(t_k) = (-1)^{k+1}\exp(\cos(x_1(t_k^-))), \\ x_2(t_k) = (-1)^{k}\exp(\sin(x_2(t_k^-))), \\ t = k\pi, k = \pm 1, \pm 2, \cdots \end{cases} \quad (17)$$

where $A(t) = \begin{pmatrix} -10 + \sin t & \cos\pi t \\ \sin\pi t & -9 - \cos t \end{pmatrix}$, $A_1(t) = \begin{pmatrix} -\sin t & -\cos\pi t \\ 0 & \cos\pi t \end{pmatrix}$, $h(t) \begin{bmatrix} \frac{1}{2}\cos t \\ \sin t \end{bmatrix}$ and $\lambda = 8$. One can verify that the conditions of Theorem 1 are satisfied. Therefore, systems (16) and (17) have an almost periodic solution.

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界条件,则变分问题的处理形式与没有内边界条件的问题相比并没有什么特殊之处,即变分问题不必作任何修改.因此,通过使用内边界条件,可以使具间断系数的边值问题的研究带来了极大的方便,能够使一些复杂方程问题简单化.

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