The CCAP-subgroups and the Finite p-supersoluble Groups*

CCAP-子群与有限 p-超可解群

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Abstract:In group G ,let p be a prime and H be a p-soluble normal subgroup of G such that G/H is p-supersoluble. If all maximal subgroups of the sylow p-subgroups of H (or $F_p(H)$ which contain $O_{p'}(H)$) are CCAP-subgroups of G, then G is p-supersoluble.

Key words:CCAP-subgroup,CCP-subgroup,CAP-subgroup, p-supersoluble, $F_p(G)$

摘要:在群 G 中,设 p 是一个素数,H 是群 G 的一个 p-可解的正规子群并使得 G/H 是 p-超可解的。若 H 的所有的 Sylow p-子群(或者 $F_p(H)$ 包含 $O_{P'}(H)$)的极大子群是 G 的 CCAP-子群,那么 G 是 p-超可解的。

关键词:CCAP-子群 CCP-子群 CAP-子群 p-超可解 p-Fitting 子群
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All groups considered in this paper are finite.

It is well-known that a supersoluble group is a group in which the chief factors are all cyclic. The class of supersoluble groups lies between nilpotent and soluble groups. In recent years many papers have investigated the supersoluble groups[1,2]. In 1993, L. M. Ezquerro gave the definition of CAP-subgroup in Reference [1].

Definition 1[1] A subgroup H of a finite group G is said to have the cover-avoiding property in G (for short, we call it a CAP-subgroup of G) if it either covers or avoids every chief factor of G.

Applying the definition, he obtained some results on the supersolubility of the group.

In Reference [2], W. Guo defined the completely conditionally permutable subgroup of G (or a CCP-subgroup in this paper).

Definition 2[2] A subgroup H is said to have completely conditionally permutable property in G (for short, a CCP-subgroup of G) if for any subgroup T of G, there exists an element $x \in \langle H,T \rangle$ such that $HT^x = T^xH$. Applying the definition, many authors investigated and curved the supersoluble and nilpotent property of the finite group G.

In this paper, Our goal is to continue these investigations and analyze the CAP-subgroup and the CCP-subgroup properties. For convenience, we give a concept of CCAP-subgroup of G in the following.

Definition 3 A subgroup H of a finite group G is said to be a CCAP-subgroup of G if it has either the completely conditionally permutable or the cover-avoiding property in G, it is either a CCP-subgroup or a CAP-subgroup of G. Hence by investigating these subgroups, we also obtain the following two main results and some corollaries.

Theorem 1 Let p be a prime, G be a group and H be a p-soluble normal subgroup of G such that G/H is p-supersoluble. If all maximal subgroups of the Sylow p-subgroups of H are CCAP-subgroups of G. Then G is p-supersoluble.

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Theorem 2 Let p be a prime, G be a group and H be a p-solvable normal subgroup of G such that G/H is p-supersoluble. If all maximal subgroups of $F_p(H)$ which contains $O_{p'}(H)$ are CCAP-subgroups of G. Then G is p- supersoluble.

And we generalize the results of L. M. Ezquerro and W. Guo. However the condition of this subgroup is weaker than the single condition. Therefore by investigating these subgroups, our works make sense. In fact, we are continueing their works.

Throughout this paper, the terminology and notations employed coincide with standard usage.

1 Some results and lemmas

In this section we collect some results and give some lemmas which are needed in the sequel.

Lemma 1 Let K be a normal subgroup and H be a subgroup of G. If H is a CCAP-subgroup of G. Then HK/K is a CCAP-subgroup of G/K.

Proof (1) If H is a CCP-subgroup of G, namely it has the completely conditionally permutable property in G. Then for any subgroup T of G, there exists an element $x \in \langle H,T \rangle$ such that $HT^x = T^xH$, since $K \trianglelefteq G$, so $KH/K \cdot (TK/K)^{xK} = KH \cdot T^{xK}K/K = T^{xK}HK/K = (TK/K)^{xK} \cdot HK/K$, thus $xK \in \langle HK/K, TK/K \rangle$. Then HK/K is a CCAP-subgroup of G/K.

(2) If H is a CAP-subgroup of G. Let $\overline{M}/\overline{N}$ be any chief factor of $\overline{G} = G/K$, we let $\overline{M} = M/K$ and $\overline{N} = N/K$, so both M and N are normal in G and $N \leqslant M$. We say M/N a chief factor of G. In fact, let H_1 be a normal subgroup of G and $N \leqslant H_1 \leqslant M$. Then $\overline{N} \leqslant \overline{H}_1 \leqslant \overline{M}$, where $\overline{H}_1 = H_1/K$. Since $\overline{M}/\overline{N}$ is any chief factor of $\overline{G} = G/K$, so $\overline{N} = \overline{H}_1$ or $\overline{H}_1 = \overline{M}$, namely $N = H_1$ or $M = H_1$. Hence M/N is a chief factor of G. By the condtion of H, $H \cap N = H \cap M$ or $HN = HM$. Then $(N/K) \cap (HK/K) = (M/K) \cap (HK/K)$ or $(N/K)(HK/K) = (M/K)(HK/K)$. So HK/K is a CAP-subgroup of G.

By (1) and (2), then HK/K is a CCAP-subgroup of G/K.

Lemma 2[3] Let M be a maximal subgroup of G and P be a normal p-subgroup of G such that $G = PM$, where p is a prime divisor of order of G. Then

(1) $P \cap M \triangleleft G$.

(2) If $p > 2$ and all minimal subgroups of P are normal in G, then $|G:M| = p$.

Lemma 3[4] G/N is supersoluble and N is a cyclic subgroup of G. Then G is supersoluble.

Lemma 4[5] If G is π-soluble, $O_{\pi'}(G) = 1$, then $C_G(O_\pi(G)) \leqslant O_\pi(G)$

Lemma 5[6] Let G be a finite group and H be a nontrivial normal subgroup of G such that $H \cap \Phi(G) = 1$. Then $F(H)$ is the direct product of all minimal normal subgroups of G which is contained in $F(H)$.

Lemma 6 Let G be a finite group and N be a normal subgroup of G. If M is a CCAP-subgroup of G, then MN is a CCAP-subgroup of G.

Proof (1) By the condition, suppose that M is a CAP-subgroup of G. Let H/K be a chief factor of G. If N covers H/K, so does NM. Suppose $H \cap N \leqslant K$, then HN/KN is a chief factor of G. G is morphic to H/K. If M covers HN/KN, then $H \leqslant HN \leqslant KNM$ and NM covers H/K. If M avoids HN/KN, $KN \cap M = HN \cap M$, then $HN \cap MN = (HN \cap M)N = (KN \cap M)N \leqslant KN$ and $MN \cap H \leqslant KN \cap H = K(N \cap H) = K$, thus MN avoids H/K.

(2) If M is a CCP-subgroup of G, then for any subgroup T of G, there exists an element $x \in \langle M,T \rangle$ such that $MT^x = T^xM$. Since $N \trianglelefteq G$, $(NM)T^x = N(MT^x) = N(T^xM) = T^x(NM)$ and $x \in \langle M,T \rangle \leqslant \langle MN,T \rangle$, then NM is a CCP-subgroup of G.

By (1) and (2), MN is a CCAP-subgroup of G.

2 Main results and proofs

Proof of theorem 1 Suppose that the theorem is false. Let G be a counterexample of minimal order. Then we have the following results.

(1) $O_{p'}(G) = 1$.

Suppose $O_{p'}(G) \neq 1$. We consider the quotient $\overline{G} = G/O_{p'}(G)$. It is clear that $\overline{G}/\overline{H} \cong G/H$ is p-supersoluble, where $\overline{H} = HO_{p'}(G)/O_{p'}(G)$. It follows that $\overline{P} = PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $\overline{H}$, where P is any Sylow p-subgroup of H. Assume that $P_1O_{p'}(G)/O_{p'}(G)$ is a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$, where P_1 is a maximal subgroup of P. By the condition of our theorem, P_1 is a CCAP-subgroup of G. By Lemma 1, we know $P_1O_{p'}(G)/O_{p'}(G)$ is a CCAP-subgroup of $G/O_{p'}(G)$. Then $G/O_{p'}(G)$ satisfies the hypotheses of our theorem. Thus, by the minimality of G, we obtain that $G/O_{p'}(G)$ is p-supersoluble, hence G is p-supersoluble, which is a contradiction.

(2) $O_p(H)$ is a minimal normal subgroup of G and $G/O_p(H)$ is p-supersoluble, $O_p(H) \not\leq \Phi(G)$ Since H is p-soluble, thus $O_p(H) \neq 1$ or $O_{p'}(H) \neq 1$. By the assertion of (1), $O_p(H) \neq 1$. Let N be a minimal normal subgroup of G which is contained in $O_p(H)$. We now consider that $\bar{G} = G/N$, clearly $\bar{G}/\bar{H} \cong G/H$ is p- supersoluble where $\bar{H} = H/N$. Let $\bar{P} = P/N \in \mathrm{Syl}_p(\bar{H})$, and $\bar{P}_1 = P_1/N$ be a maximal subgroup of $\bar{P}$, where P_1 is a maximal subgroup of P. Since P_1 is a CCAP-subgroup of G, by Lemma 1, $\bar{P}_1$ is a CCAP-subgroup of $\bar{G}$. Thus $\bar{G}$ satisfies the hypotheses of our theorem, by the minimality of G, $\bar{G}$ is p-supersoluble. Since the class of all p-supersoluble groups is a saturated formation, we know that N is the unique minimal normal subgroup of G which is contained in $O_p(H)$ and N isn't contained in $\Phi(G)$. Therefore there exists a maximal subgroup M of G such that $G = MN$ and $M \cap N = 1$. According to Lemma 2, $O_p(H) \cap M \trianglelefteq G$, so $O_p(H) \cap M = 1$ and $N = O_p(H)$.

(3) By the result of (2), $N = O_p(H)$ is the unique minimal normal subgroup of G. So there exists a maximal subgroup of G such that $G = NM$ and $M \cap N = 1$. Hence any Sylow p-subgroup of G has the form of M_pN, where M_p is a Sylow p-subgroup of M. Let G_p be any Sylow p-subgroup of G and $G_p = M_pN$.

Suppose that H_1 is any maximal subgroup of G_p which contains M_p. Let P be any Sylow p-subgroup of H. Thus, $PH_1 \leqslant G_p$. Since H_1 is a maximal subgroup of G_p, $PH_1 = H_1$ or $PH_1 = G_p$. If $PH_1 = H_1$, so $P \leqslant H_1$, it follows that $N \leqslant P \leqslant H_1$. Then $G_p = M_pN \leqslant H_1$, a contradiction. Hence $PH_1 = G_p$, clearly, $P_1 = P \cap H_1$ is a maximal subgroup of P. By the condition of our theorem, suppose that P_1 is a CCP-subgroup of G. So there exists an element $x \in \langle P_1, M \rangle$ such that $P_1M^x = M^xP_1$. By the maximality of M, $P_1M^x = G$ or $P_1M^x = M^x$. If $P_1M^x = G$, by $G = NM$ and $M \cap N = 1$, so $x = mn$ where $m \in M$ and $n \in N$. So $P_1M^x = P_1M^{mn} = P_1M^n = (P_1^{n^{-1}}M)^n = P_1M = G$, then $G = H_1M$. Since $H_1 = H_1 \cap G_p = H_1 \cap M_pN = M_p(H_1 \cap N)$, $G = H_1M = M(H_1 \cap N) = MN$. Then $H_1 \cap N = N$, we obtain that $N \leqslant H_1$. It follows that $G_p = M_pN \leqslant H_1$, a contradiction. Then $P_1M^x = M^x$. Clearly, $P_1 \leqslant M^x$. Again, since $M^x \cap N = M \cap N = 1$, thus $P_1 \cap N = 1$. Hence $P = P_1N$, we obtain $|P:P_1| = |N| = p$. So N is a cyclic subgroup of order p. By Lemma 3, G is p-supersoluble, a contradiction. Futhermore, P_1 isn't a CCP-subgroup of G but a CAP-subgroup of G. Namely P_1 is a CAP-subgroup of G. Then by the minimality of N, P_1 covers or avoids $N/1$. If P_1 avoids $N/1$, $P_1 \cap N = P_1 \cap 1 = 1$, then $|P:P_1| = |N| = p$, N is a cyclic subgroup of order p. By the result of (2) and Lemma 3, G/N is p-supersoluble, then G is p- supersoluble, a contradiction. So we suppose that P_1 covers $N/1$. So $P_1N = P_1$. Again $N \leqslant P_1 \leqslant H_1$, thus $G_p = M_pN \leqslant H_1$, which is a contradiction. This is the final contradiction.

The proof is completed.

Corollary 1 (Reference [1], Theorem A) Let p be a prime, G be a group and H be a p- soluble normal subgroup of G such that G/H is p- supersoluble. If all maximal subgroups of the Sylow p- subgroups of H are CAP-subgroups of G. Then G is p-supersoluble.

Corollary 2 Let p be a prime, G be a group and H be a p- soluble normal subgroup of G such that G/H is p- supersoluble. If all maximal subgroups of the Sylow p-subgroups of H are CCP-subgroups of G. Then G is p-supersoluble.

Next, we shall gerenalize the theorem. We consider the p-Fitting subgroup of G and write $F_p(G)$. In fact, $F_p(G) = O_{p'p}(G)$ (Reference [5], P262.)

Proof of theorem 2 Suppose that the theorem is false and let G be a counterexample of minimal order. Then

(1) $O_{p'}(G) = 1$.

Firstly we show $O_{p'}(H) = 1$. Otherwise, $T = O_{p'}(H) \neq 1$. We consider the quotient $\bar{G} = G/T$. Clearly, $\bar{G}/\bar{H} \cong G/H$ is p- supersoluble, where $\bar{H} = H/T$. Then $O_{p'}(\bar{H}) = 1$ and $F_p(\bar{H}) = F_p(H)/T$. Let M/T be a maximal subgroup of $F_p(\bar{H})$, then $T \leqslant M \leqslant F_p(H)$ and M is a maximal subgroup of $F_p(H)$. Since M is a CCAP-subgroup of G, by Lemma 1, M/T is a CCAP-subgroup of G/T. Hence $\bar{G}$ satisfies the condition of our theorem, by the minimality of $|G|$, $\bar{G}$ is p-supersoluble. So is G, a contradiction.

Now we show that $O_{p'}(G) = 1$. Otherwise, suppose that $R = O_{p'}(G) \neq 1$. We consider the quotient $\overline{G} = G/R$. It is clear that $\overline{G}/\overline{H} \cong G/HR$ is p-supersoluble, where $\overline{H} = HR/R$. Applying the condition $O_{p'}(H) = 1$, we obtain $F_p(H) = O_p(H)$, so $F_p(\overline{H}) = F_p(H)R/R$. Let $\overline{P}_1 = P_1R/R$ be a maximal subgroup of $F_p(\overline{H})$. Also let P_1 be a maximal subgroup of $F_p(H)$. Since P_1 is a CCAP-subgroup of G, by Lemma 1, P_1R/R is a CCAP-subgroup of G/R. By the minimality of $|G|$, we obtain $\overline{G}$ is p-supersoluble, it follows that G is p- supersoluble, a contradiction.

(2) $H \cap \Phi(G) = 1$.

For convenience, write $L = H \cap \Phi(G)$. If $L \neq 1$, we consider the quotient $\overline{G} = G/L$. By Reference [7] (Hu, Ⅲ, 3.5), $F(H/L) = F(H)/L$, so $F(H/L) = O_p(H)/L$. Moreover, let $K/L = O_{p'}(H/L)$ and S be a Hall p'-subgroup of K, then there is $K = SL$ and by Arguement of Frattini, $G = KN_G(S) = LN_G(S) = N_G(S)$. thus S is normal in G. So $S = 1$ hence $O_{p'}(H/L) = 1$. We obtain $F_p(H/L) = O_p(H/L) = O_p(H)/L = F_p(H)/L$. If P_1/L is a maximal subgroup of $F_p(H/L)$, then P_1 is a maximal subgroup of $F_p(H)$. Since P_1 is a CCAP-subgroup of G, by Lemma 1, P_1/L is a CCAP-subgroup of G/L. The minimality of $|G|$ implies that $\overline{G}$ is p-supersoluble, it follows that G is p-supersoluble, a contradiction.

(3) The final contradiction.

Since H is p-soluble and $O_{p'}(G) = 1$, by Lemma 4, $C_H(O_p(H)) \leqslant O_p(H)$. Since $\Phi(H) = 1$, by Reference [7] (Hu1, Ⅲ, 4.5), $F(H) = O_p(H)$ is a nontrivial elementary abelian p-group, hence $C_H(F(H)) = F(H)$.

Now we show that all minimal normal subgroups of G which are contained in $O_p(H)$ are cyclic.

Suppose that all minimal normal subgroups of G contained in $O_p(H)$ are noncyclic.

Let N be a minimal subgroup of G which is contained in $O_p(H)$. Since $H \cap \Phi(G) = 1$, So there exists a maximal subgroup of G such that $G = NM$ and $M \cap N = 1$. Hence any Sylow p-subgroup of G has the form of M_pN, where M_p is a Sylow p-subgroup of M. Let G_p be any Sylow p-subgroup of G and $G_p = M_pN$. Suppose that H_1 is any maximal subgroup of G_p which contains M_p. Thus, clearly, $P_1 = O_p(H) \cap H_1$ is a maximal subgroup of $O_p(H)$. By the condition of our theorem, suppose that P_1 is a CCP-subgroup of G. So there exists an element $x \in \langle P_1, M \rangle$ such that $P_1M^x = M^xP_1$. By the maximality of M, $P_1M^x = G$ or $P_1M^x = M^x$. If $P_1M^x = G$, then $G = H_1M$. Since $H_1 = H_1 \cap G_p = H_1 \cap M_pN = M_p(H_1 \cap N)$, $G = H_1M = M(H_1 \cap N) = MN$. Then $H_1 \cap N = N$, we obtain that $N \leqslant H_1$. It follows that $G_p = M_pN \leqslant H_1$, a contradiction. Then $P_1M^x = M^x$. Clearly, $P_1 \leqslant M^x$. Again, since $M^x \cap N = M \cap N = 1$, $P_1 \cap N \doteq 1$. Hence $O_p(H) = P_1N$, we obtain $|O_p(H):P_1| = |N| = p$. So N is a cyclic subgroup of order p, a contradiction. Futhermore, P_1 isn't a CCP-subgroup of G but a CAP-subgroup of G. Namely P_1 is a CAP-subgroup of G. Then by the minimality of N, P_1 covers or avoids $N/1$. If P_1 avoids $N/1$, $P_1 \cap N = P_1 \cap 1 = 1$, then $|O_p(H):P_1| = |N| = p$, N is a cyclic subgroup of order p, a contradiction. So we suppose that P_1 covers $N/1$. So $P_1N = P_1$, it follows that $N \leqslant P_1 \leqslant H_1$. Thus $G_p = M_pN \leqslant H_1$, a contradiction.

Thus we show that all minimal normal subgroups of G contained in $O_p(H)$ are cyclic.

Finally, by Lemma 5, $F(H) = N_1 \times \cdots \times N_r$, where N_i is a normal subgroup of G of order p. For every positive integer i, the quotient $G/C_G(N_i) \lesssim Aut(N_i)$, so $G/C_G(N_i)$ is abelian. Since G/H is p-supersoluble, $G/(H \cap C_G(N_i)) = G/C_H(N_i)$ is p-supersoluble. Then $G/\bigcap_{i=1}^{r}(C_G(N_i)))$ is p-supersoluble. In fact, we obtain that $G/F(H)$ is p-supersoluble. Since $\bigcap_{i=1}^{r}(C_G(N_i))) = C_H(F(H)) = F(H)$. But all chief factors of G which are contained in $F(H)$ are cylcic groups of order p, so G is p-supersoluble. This is the final contradiction. The proof is completed.

Corollary 3 (Reference [1], Ththeorem B) Let p be a prime, G be a group and H be a p-soluble normal subgroup of G such that G/H is p-supersoluble. If all maximal subgroups of $F_p(H)$ which contains $O_{p'}(H)$ are CAP-subgroups of G. Then G is p-supersoluble.

Corollary 4 Let p be a prime, G be a group and H be a p-soluble normal subgroup of G such that G/H is p-supersoluble. If all maximal subgroups of $F_p(H)$ which contains $O_{p'}(H)$ are CCP-subgroups of G. Then G is p-supersoluble.

Corollary 5 Let p be a prime, G be a group and H be a p-soluble normal subgroup of G such that G/H is p- supersoluble. If all maximal subgroups of the Sylow p-subgroups of $F_p(H)$ are CCAP-subgroups of G. Then G is p-supersoluble.

Proof Since $F_p(H)$ is p-nilpotent, thus all maximal subgroups of $F_p(H)$ which contains $O_{p'}(H)$ have the form of $P_1O_{p'}(H)$, where P_1 is a maximal subgroup of some Sylow p-subgroup of $F_p(H)$. Since P_1 is a CCAP-subgroup of G, by Lemma 6, $P_1O_{p'}(H)$ is a CCAP-subgroup of G. By the theorem 2, we obtain that G is p-supersoluble. The proof is completed.

Corollary 6 Let p be a prime, G be a group and H be a p-soluble normal subgroup of G such that G/H is p-supersoluble. If all maximal subgroups of the Sylow p-subgroups of $F_p(H)$ are CCP-subgroups of G. Then G is p-supersoluble.

Corollary 7 Let p be a prime, G be a group and H be a p-soluble normal subgroup of G such that G/H is p-supersoluble. If all maximal subgroups of the Sylow p-subgroups of $F_p(H)$ are CAP-subgroups of G. Then G is p-supersoluble.

References:

[1] EZQUERRO L M. A contribution to theory of finite supersolvable groups[J]. Rend Sem Mat Unver Pradova, 1993, 89: 161-170.

[2] GUO WENBIN, SHUM K P, ALEXANDER SKIBA. Criterions of supersolublity for products of supersoluble groups[J]. Siberian Math, 2004, 45(1): 115-128.

[3] WANG Y, WEI H, LI Y. A generalization of Kramer's theorem and its applications[J]. Bull Austral Math Soc, 2002, 65: 467-475.

[4] ZHANG Y. Translation, Between nilpotence and solvable [M]. Wuhan: Wuhan University Publishing Company, 1988.

[5] ROBINSON D J S. A course in the theory of groups [M]. New York, Berlin: Springer-Verlag, 1980.

[6] LI Y, WANG Y, WEI H. The influence of π-quasinormality of some subgroups of a finite group[J]. Arch Math, 2003, 81(3): 245-252.

[7] HUPPERT B. Endliche gruppen I[M]. New York: Springer-Verlag, 1967.

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