Oscillation of Higher-order Nonlinear Neutral Differential Equation*

高阶非线性中立型微分方程的振动性

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Abstract: The oscillation of a kind of higher-order nonlinear neutral differential equation

$$\{a(t,x(t))[x(t)+\sum_{i=1}^{m}c_i(t)x(\tau_i(t))]^{(n-1)}\}'+\int_a^b F(t,\zeta,x(g(t,\zeta)))\mathrm{d}\sigma(\zeta)=0$$

(Where $t>t_0$ and $n\geqslant 2$ is even) is discussed. Some sufficient conditions for the oscillation of the above equation are obtained.

Key words: differential equation, oscillation, continuous distributed delay, nonlinear

摘要:研究一类高阶非线性中立型方程

$$\{a(t,x(t))[x(t)+\sum_{i=1}^{m}c_i(t)x(\tau_i(t))]^{(n-1)}\}'+\int_a^b F(t,\zeta,x(g(t,\zeta)))\mathrm{d}\sigma(\zeta)=0$$

(其中 $t>t_0$,$n\geqslant 2$ 为偶数)的振动性,并获得该方程振动的一些充分条件.

关键词:微分方程　振动性　连续分布时滞　非线性

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Recently, there are many papers concerning the oscillation of second order neutral differential equation[1~8]. However, only a few papers investigate the oscillation of higher-order nonlinear neutral differential equation with continuous distributed delay[5~8].

Fu and Liu[5] gave some sufficient conditions for the oscillation of the equation

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}[y(t)+\lambda(t)y(t-\tau)]+\int_a^\beta Q(t,\zeta)F(y[h(t,\zeta)])\mathrm{d}\sigma(\zeta)=0.$$

A. Zafer[6] showed that the equation

$$[y(t)+Q(t)y(\tau(t))]^{(n)}+f(t,x(t),x(\sigma(t)))=0$$

oscillates if $\phi(t)$ is a nonnegative continuous function on $[0,+\infty]$ and that $w(t)>0$ for $t>0$ is continuous and non-decreasing on $[0,+\infty]$ with

$$\|f(t,x,y)\|\geqslant \phi(t)w(\frac{\|y\|}{[1-a(\sigma(t))][\sigma(t)]^{n-1}}),$$

and

$$\int_0^{\pm\lambda}\frac{\mathrm{d}x}{w(x)}<\infty$$

for every $\lambda>0$. If n is even and

$$\int_0^\infty\phi(t)\mathrm{d}t=\infty.$$

In this paper, we consider a more general higher-order nonlinear neutral differential equation

$$\{a(t,x(t))[x(t)+\sum_{i=1}^{m}c_i(t)x(\tau_i(t))]^{(n-1)}\}'+\int_a^b F(t,\zeta,x(g(t,\zeta)))\mathrm{d}\sigma(\zeta)=0,\quad(1)$$

which has continuous distributed delay. And we will give some sufficient conditions for the oscillation of Equation (1).

1 Preliminaries

Definition 1 A solution $x(t)$ of Equation (1) is called eventually positive (or eventually negative) if there exists a constant $T_0>t_0$, such that $x(t)>0$ (or

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$x(t) < 0$) for $t > T_0$.

Definition 2 A solution of Equation(1) is called oscillatory if the solution $x(t)$ is not eventually positive and not eventually negative.

Definition 3 Equation(1) is called oscillatory if all solutions of Eq. (1) are oscillatory.

Throughout this paper, we always assume that

(H_1) $c_i(t), \tau_i(t) \in ([t_0, +\infty), [0, +\infty)), \tau_i(t) \leqslant t, \lim_{t\to+\infty} \tau_i(t) = +\infty, i = 1,2,\cdots,m$;

(H_2) $g(t,\zeta) \in ([t_0, +\infty) \times [a,b], [0, +\infty))$, $g(t,\zeta) \leqslant t, t \in [t_0, +\infty), \zeta \in [a,b]$, the function $g(t,\zeta)$ is non-decreasing with respect to t and ζ, respectively, and $\lim_{t\to+\infty} \min_{a\leqslant\zeta\leqslant b} g(t,\zeta) = +\infty$;

(H_3) $F(t,\zeta,x) \in C([t_0, +\infty) \times [a,b] \times R, R), \sigma(\zeta) \in C([a,b],R)$, the function $\sigma(\zeta)$ is non-decreasing, and the integral of Equation(1) is Stieltjes integral.

2 Main results

Lemma 1[9] If $u(t)$ is n-times differential function on $[0, +\infty]$ of constant sign, $u^{(n)}(t)$ is of constant sign and identically zero in any interval $[t_0, +\infty]$, and $u^{(n)}(t)u(t) \leqslant 0$, then there exists an integer $l, 0 \leqslant l \leqslant n-1$ with $n-1$ odd, such that for $t \geqslant t_0$,

$u^{(k)}(t) > 0, 0 \leqslant k \leqslant l$;

$u(t)u^{(k)}(t) > 0, k = 0,1,\cdots,l$;

$(-1)^{k-1}u(t)u^{(k)}(t) > 0, k = l,\cdots,n-1$.

Theorem 1 Assume that the following conditions hold.

(H_4) $a(t,x(t)) \in C([t_0, +\infty) \times R, (0, +\infty))$;

(H_5) $0 \leqslant \sum_{i=1}^{m} c_i(t) \leqslant 1, t \geqslant t_0$;

(H_6) There exist two functions $q(t,\zeta) \in C([t_0, +\infty) \times [a,b], [0, +\infty))$ and $f(x) \in C(R,R)$, such that

$$F(t,\zeta,x)sgnx \geqslant q(t,\zeta)f(x)sgnx, \quad (2)$$

$$-f(-x) \geqslant f(x) \geqslant \lambda x, \quad (3)$$

where $x > 0$, and λ is a positive constant;

(H_7) There exists a non-decreasing function $\phi(t)$ for $t \geqslant t_0$, such that $0 < \phi(t) \leqslant a(t,x(t))$, then

$$\int_{t_0}^{+\infty} \frac{1}{\phi(s)} ds = \infty;$$

(H_8) If $\frac{dg}{dt}(t,\zeta)$ exists for all $t > 0$, and there exists a monotonically increasing function $\varphi(t) \in C'[(t_0, +\infty), (0, +\infty)]$, such that

$$\int_{t_0}^{+\infty} \{\lambda\varphi(s)\int_a^b q(s,\zeta)[1 - \sum_{i=1}^{m} c_i(g(s,\zeta))]d\sigma(\zeta) - r\varphi'(s)\}ds = +\infty, \quad (4)$$

for any number $r > 0$. Then Equation (1) is oscillatory.

Proof Assume that $x(t)$ is a non-oscillatory solution of Equation(1). Without loss of generality, we assume that $x(t)$ is eventually positive(the proof is similar when $x(t)$ is eventually negative). For the sake of convenience, the function $y(t)$ is defined by

$$y(t) = x(t) + \sum_{i=1}^{m} c_i(t)x(\tau_i(t)). \quad (5)$$

By the conditions (H_1) and (H_2), there exists a $t_1 \geqslant t_0$ such that $x(\tau_i(t)) > 0, x(g(t,\zeta)) > 0, t \geqslant t_1, \zeta \in [a, b], i = 1,2,\cdots,m$, so we obtain

$$y(t) > 0, t \geqslant t_1. \quad (6)$$

Using Inequalities (2) and (3), from Equation(1) it follows that

$$[a(t,x(t))y^{(n-1)}(t)]' = -\int_a^b F(t,\zeta,x(g(t,\zeta)))d\sigma(\zeta) \leqslant -\lambda\int_a^b q(t,\zeta)x(g(t,\zeta))d\sigma(\zeta) \leqslant 0, t \geqslant t_1. \quad (7)$$

Therefore, the function $a(t,x(t))y^{(n-1)}(t)$ is monotonically decreasing. In the following, we can proof that $y^{(n-1)}(t) \geqslant 0$ for $t \geqslant t_1$. In fact, if there exists a $t_2 \geqslant t_1$, such that

$$y^{(n-1)}(t_2) < 0.$$

Integrating both sides of Inequality (7) from t_2 to t, by the condition (H_1), we have

$a(t,x(t))y^{(n-1)}(t) \leqslant a(t_2,x(t_2))y^{(n-1)}(t_2) = L < 0$, and hence

$$y^{(n-1)}(t) \leqslant \frac{L}{a(t,x(t))}. \quad (8)$$

Integrating both sides of Inequality (8) from t_2 to t, we get

$$y^{(n-2)}(t) \leqslant y^{(n-2)}(t_2) + \int_{t_2}^{t} \frac{L}{a(s,x(s))} ds.$$

Let $t \to +\infty$, by the condition (H_7), we have

$$\lim_{t\to+\infty} y^{(n-2)}(t) = -\infty,$$

and hence

$$\lim_{t\to+\infty} y(t) = -\infty.$$

This contradicts Inequality (6). Therefore we have

$$y^{(n-1)}(t) \geqslant 0 \text{ for } t \geqslant t_1.$$

By Lemma 1, there exists a $t_3 \geqslant t_2$ and an odd l, such that

$$y^{(i)}(t) > 0, 0 \leqslant i \leqslant l, t \geqslant t_3; (-1)^{k-1} y^{(k)}(t) > 0, k = l, \cdots, n-1, t \geqslant t_3;$$

let $i = 1$,we get

$$y'(t) > 0, t \geqslant t_3.$$

In view of $y(t) \geqslant x(t)$ and the monotone of the function $y(t)$, together with Inequality (7) and the condition (H_1), we obtain

$$0 \geqslant [a(t,x(t))y^{(n-1)}(t)]' + \lambda \int_a^b q(t,\zeta) x(g(t,\zeta)) d\sigma(\zeta) \geqslant [a(t,x(t))y^{(n-1)}(t)]' + \lambda \int_a^b q(t,\zeta)[y(g(t,\zeta)) - \sum_{i=1}^m c_i(g(t,\zeta)) y(\tau_i(g(t,\zeta)))] d\sigma(\zeta)$$
$$\geqslant [a(t,x(t))y^{(n-1)}(t)]' + \lambda \int_a^b q(t,\zeta)[1 - \sum_{i=1}^m c_i(g(t,\zeta))] y(g(t,\zeta)) d\sigma(\zeta). \quad (9)$$

Since the function $g(t,\zeta)$ is non-decreasing, using the condition (H_2), we have

$$g(t,a) \leqslant g(t,\zeta), t \geqslant t_3, \zeta \in [a,b].$$

By Inequality (9) and the monotone of the function $y(t)$,we have

$$[a(t,x(t))y^{(n-1)}(t)]' + \lambda y(g(t,a)) \int_a^b q(t,\zeta)[1 - \sum_{i=1}^m c_i(g(t,\zeta))] d\sigma(\zeta) \leqslant 0. \quad (10)$$

Define

$$w(t) = \varphi(t) \frac{a(t,x(t))y^{(n-1)}(t)}{y(g(t,a))},$$

then

$$w(t) > 0, t \geqslant t_3. \quad (11)$$

Since $a(t,x(t))y^{(n-1)}(t)$ is decreasing and the function $\varphi(t)$ is monotonically increasing, from Inequality (10) and the existence of $\frac{dg}{dt}(t,a)$,it follows that

$$w'(t) = \phi'(t) \frac{a(t,x(t))y^{(n-1)}(t)}{y(g(t,a))} + \varphi(t) \frac{[a(t,x(t))y^{(n-1)}(t)]'}{y(g(t,a))} - \frac{\varphi(t) a(t,x(t)) y^{(n-1)}(t) y'(g(t,a)) g'(t,a)}{y^2(g(t,a))} \leqslant$$
$$\phi'(t) \frac{a(t,x(t))y^{(n-1)}(t)}{y(g(t,a))} + \varphi(t) \frac{[a(t,x(t))y^{(n-1)}(t)]'}{y(g(t,a))} \leqslant \phi'(t) \frac{a(T,x(T))y^{(n-1)}(T)}{y(g(T,a))} - \lambda\varphi(t) \int_a^b q(t,\zeta)[1 - \sum_{i=1}^m c_i(g(t,\zeta))] d\sigma(\zeta),$$

where $T \geqslant t_3$, such that $\frac{dg}{dt}(t,a) > 0$ for $t \geqslant T$.

Let $r = \frac{a(T,x(T))y^{(n-1)}(T)}{y(g(T,a))} > 0$,then

$$w'(t) \leqslant -\{\lambda\varphi(t) \int_a^b q(t,\zeta)[1 - \sum_{i=1}^m c_i(g(t,\zeta))] d\sigma(\zeta) - r\phi'(t)\}, t \geqslant T.$$

Integrating both sides of the above inequality from T to t ,we get

$$w(t) \leqslant w(T) - \int_T^t \{\lambda\varphi(s) \int_a^b q(s,\zeta)[1 - \sum_{i=1}^m c_i(g(s,\zeta))] d\sigma(\zeta) - r\phi'(s)\} ds.$$

Let $t \to \infty$, from Equation (4), we know that this contradicts Inequality (11). Hence, the proof of Theorem 1 is completed.

Corollary 1 In theorem 1, if we let $\varphi(t) \equiv 1$, and if $\int_{t_0}^{\infty} \int_a^b q(s,\zeta)[1 - \sum_{i=1}^m c_i(g(s,\zeta)) d\sigma(\zeta) ds \equiv \infty$. Then Equation(1) is oscillatory.

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