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Sub-Hom-coassociative Coalgebra
子 Hom-余结合余代数

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Abstract According to the notion of Hom-coassociative coalgebra and Sub-Hom-coassociative coalgebra, we investigate some fundamental properties of them, finally obtain the close relationships between the Sub-Hom-coassociative coalgebra and the ideal of the dual Hom-associative algebra.

Key words Sub-Hom-coassociative coalgebra, Hom-associative algebra, homomorphism

摘要:根据 Hom-余结合余代数及子 Hom-余结合余代数的相关概念,讨论子 Hom余结合余代数的基本性质,得到子 Hom余结合余代数与对偶 Hom结合代数的理想之间的相互关系.

关键词:子 Hom-余结合余代数 Hom结合余代数 同态
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A Hom-algebra structure is a multiplication on a linear space where the structure is twisted by a homomorphism. Hom-associative algebras which were introduced by Makhlouf and Silvestrov in [1], generalize the well known associative structures to a situation where associativity law is twisted. Based on these, we obtain the Hom-coassociative coalgebra structures by dualization, then define the new notion of Sub-Hom-coassociative coalgebras and discuss some fundamental properties of them.

Throughout this paper, we always assume that K is an algebraically closed field of characteristic 0 and V is a linear space over K, $V^* = \mathrm{Hom}_K(V,K)$ denotes the dual space of V. If $S \subseteq V$ is the subset of V, then $S^{\perp} = \{v^* \in V^* \mid \langle v^*, S\rangle = 0\}$. If $T \subseteq V^*$ is the subset of V^*, then $T^{\perp} = \{v \in V \mid \langle T, v\rangle = 0\}$.

1 Hom-associative algebra

Definition 1.1[1] A Hom-associative algebra is a quadruple $(V, \_, T, Z)$ together with three K-linear maps

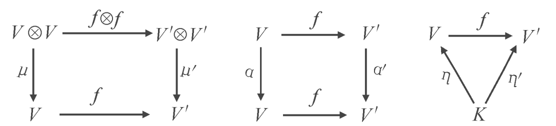
$$\_: V \otimes V \to V,\ T: V \to V,\ Z: K \to V$$

which satisfy the following conditions:

(1.1) $\_ \circ (\_ \otimes T) = \_ \circ (T \otimes \_)$;

(1.2) $\_ \circ (Z \otimes id) = id$ and $\_ \circ (id \otimes Z) = id$.

Definition 1.2 Let $(V, \_, T, Z)$ and $(V', \_', T', Z')$ be two Hom-associative algebras. A linear map $f: V \to V'$ is a homomorphism of Hom-associative algebras if the following diagrams are commutative.



Proposition 1.1 Suppose $(V, \_, T, Z)$ is a Hom-associative algebra, I is an ideal of V and $c: V \to V/I$ is the natural map onto the quotient vector space. If $T(I) \subseteq I$, then V/I has a unique Hom-associative algebra structure such that c is a morphism of Hom-associative algebras.

Proof Let $V/I = E$. Firstly, we find $\__E, T_E, Z_E$, $\forall a \otimes b \in \mathrm{Ker}(c \otimes c)$,

$$(c \otimes c)(a \otimes b) = c(a) \otimes c(b) = 0,$$

then $c(a) = 0$ or $c(b) = 0$, i.e. $a \in I$ or $b \in I$. But I is an ideal, so $c\_(a \otimes b) = c(ab) = 0$, i.e. $a \otimes b \in$

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$\mathrm{Ker}\,\pi\mu$, thus $\mathrm{Ker}(\pi\otimes\pi)\subseteq \mathrm{Ker}\,\pi\mu$. According to the universe properties of module, there is a unique K-map μ_E: $E\otimes E\to E$ s.t.

$$\mu_E(\pi\otimes\pi) = \pi\mu. \qquad (1)$$

Let

$$\eta_E = \pi\eta: K\to E \qquad (2)$$

By (1), $\forall a,b\in V$, $\pi(ab) = \pi\mu(a\otimes b) = \mu_E(\pi\otimes\pi)(a\otimes b) = \mu_E(\pi(a)\otimes\pi(b)) = \pi(a)\pi(b)$.

Since $\pi\alpha(\mathrm{Ker}\,\pi) = \pi\alpha(I)\subseteq \pi(I) = 0$, then $\mathrm{Ker}\,\pi\subseteq \mathrm{Ker}\,\pi\alpha$. According to the universe properties of module, there is a unique K-map α_E: $E\to E$, such that

$$\pi\alpha = \alpha_E\pi. \qquad (3)$$

Then we need to pove that $(E,\mu_E,\alpha_E,\eta_E)$ is a Hom-associative algebra.

$\forall \bar a,\bar b,\bar c\in E$, there exist $a,b,c\in V$ s.t. $\pi(a)=\bar a$, $\pi(b)=\bar b$, $\pi(c)=\bar c$. Since

$\mu_E(\alpha_E\otimes\mu_E)(\bar a\otimes\bar b\otimes\bar c) = \mu_E(\alpha_E\pi(a)\otimes\pi(b)\pi(c)) = \pi\alpha(a)\pi(b)\pi(c) = \pi(\alpha(a)\mu(b\otimes c)) = \pi\mu(\alpha(a)\otimes\mu(b\otimes c)) = \pi\mu(\alpha\otimes\mu)(a\otimes b\otimes c) = \pi\mu(\mu\otimes\alpha)(a\otimes b\otimes c) = \pi\mu(ab\otimes\alpha(c)) = \pi(ab\alpha(c)) = \bar a\bar b\pi\alpha(c) = \bar a\bar b\alpha_E\pi(c) = \mu_E(\mu_E(\bar a\otimes\bar b)\otimes\alpha_E(\bar c)) = \mu_E(\mu_E\otimes\alpha_E)(\bar a\otimes\bar b\otimes\bar c)$.

Thus $\mu_E(\alpha_E\otimes\mu_E) = \mu_E(\mu_E\otimes\alpha_E)$.

$\forall k\otimes\bar a\in K\otimes E$, there exists $a\in A$, s.t. $\pi(a) = \bar a$. Since

$\mu_E(\eta_E\otimes id)(k\otimes\bar a) = \mu_E(\eta_E(k)\otimes\pi(a)) = \mu_E(\pi\eta(k)\otimes\pi(a)) = k\mu_E(\pi(1)\otimes\pi(a)) = k\bar 1\bar a = k\otimes\bar a$,

So $\mu_E(\eta_E\otimes id) = id$. Similarly, $\mu_E(id\otimes\eta_E) = id$. Therefore $(E,\mu_E,\alpha_E,\eta_E)$ is a Hom-associative algebra. According to (1), (2) and (3), we know π is a morphism of Hom-associative algebras.

Theorem 1.1 Let $(A,\mu_A,\alpha_A,\eta_A)$ and $(B,\mu_B,\alpha_B,\eta_B)$ be Hom-associative algebras, f: $A\to B$ is a morphism of Hom-associative algebras, I is an ideal of A and π: $A\to A/I$ is the natural map. If $\alpha_A(I)\subseteq I$, $\mathrm{Ker}\,\pi\subseteq \mathrm{Ker} f$, then there is a unique morphism of Hom-associative algebras g: $A/I\to B$, s.t. $g\pi = f$.

Proof Let $E = A/I$. By Proposition 2.1, $(E,\mu_E,\alpha_E,\eta_E)$ is a Hom-associative algebra where $\mu_E(\pi\otimes\pi) = \pi\mu_A$, $\pi\alpha_A = \alpha_E\pi$, $\eta_E = \pi\eta_A$.

Since π is surjective and $\mathrm{Ker}\,\pi\subseteq \mathrm{Ker} f$, According to the universe properties of module, there is a unique K-map g: $E\to B$, s.t. $g\pi = f$.

Firstly, $\forall a,b\in A$,

$\mu_B(g\otimes g)(\pi\otimes\pi)(a\otimes b) = \mu_B(g\pi\otimes g\pi)(a\otimes b) = \mu_B(g\pi(a)\otimes g\pi(b)) = \mu_B(f(a)\otimes f(b)) = \mu_B(f\otimes f)(a\otimes b) = f\mu_A(a\otimes b) = g\pi\mu_A(a\otimes b) = g\mu_E(\pi\otimes\pi)(a\otimes b)$,

Thus $\mu_B(g\otimes g) = g\mu_E$.

Secondly, $\forall a\in A$, $\alpha_B g\pi(a) = \alpha_B f(a) = f\alpha_A(a) = g\pi\alpha_A(a) = g\alpha_E\pi(a)$, Thus $\alpha_B g = g\alpha_E$.

Finally, $\forall k\in K$, $g\eta_E(k) = g\pi\eta_A(k) = f\eta_A(k) = \eta_B(k)$, Thus $g\eta_E = \eta_B$.

Therefore g is a morphism of Hom-associative algebras.

2 Hom-coassociative coalgebra and Sub-Hom-coassociative coalgebra

Definition 2.1 A Hom-coassociative coalgebra is a quadruple $(V,\Delta,\alpha,\varepsilon)$ together with three K-linear maps

$$\Delta: V\to V\otimes V,\ \alpha: V\to V,\ \varepsilon: V\to K$$

which satisfy the conditions:

(2.1) $(\alpha\otimes\Delta)\circ\Delta = (\Delta\otimes\alpha)\circ\Delta$

(2.2) $(id\otimes\varepsilon)\circ\Delta = id$ and $(\varepsilon\otimes id)\circ\Delta = id$

Remark 2.1 Let $(V,\Delta,\alpha,\varepsilon)$ be a Hom-coassociative coalgebra. We introduce the Sweedler notation[2,3] $\Delta(c) = \sum c_1\otimes c_2$, for any $c\in V$.

Duality by Definition 1.2, we obtain

Definition 2.2 Let $(V,\Delta,\alpha,\varepsilon)$ and $(V',\Delta',\alpha',\varepsilon')$ be two Hom-coassociative coalgebras. A linear map g: $V\to V'$ is a homomorphism of Hom-coassociative coalgebras, if it satisfies

$$(g\otimes g)\circ\Delta = \Delta'\circ g,\ g\circ\alpha = \alpha'\circ g \text{ and } \varepsilon = \varepsilon'\circ g.$$

Definition 2.3 Suppose $(V,\Delta,\alpha,\varepsilon)$ is a Hom-coassociative coalgebra and W is a subspace which satisfies the conditions $\Delta(W)\subseteq W\otimes W$ and $\alpha(W)\subseteq W$. We can check that $(W,\Delta|_W,\alpha_W,\varepsilon_W)$ is also a Hom-coassociative coalgebra, and W is called a Sub-Hom-coassociative coalgebra of V.

Remark 2.2 Let $(V,\Delta,\alpha,\varepsilon)$ be a Hom-coassociative coalgebra, then

(i) $G(V) = \{x\in V \mid \Delta(x) = x\otimes x, \alpha(x) = x, \varepsilon(x) = 1\}$ is a Sub-Hom-coassociative coalgebra of V.

(ii) Any nonzero element $x\in G(V)$ fixes a one dimensional Sub-Hom-coassociative coalgebra Kx.

Remark 2.3 Suppose $(V,\Delta,\alpha,\varepsilon)$ is a Hom-coassociative coalgebra with structure maps $\Delta(x) = x\otimes x$, $\alpha(x) = x$ and $\varepsilon(x) = 1$, then $(V,\Delta,\alpha,\varepsilon)$ can be represented as the sum of Sub-Hom-coassociative coalgebras of V.

Proposition 2.1 If $g: C\rightarrow D$ is a homomorphism of Hom-coassociative coalgebras, then Img is a Sub-Hom-coassociative coalgebra of D.

Proof We have $\Delta_D\circ g=(g\otimes g)\circ\Delta_C$ and $U_D\circ g=g\circ U_C$ because g is a homomorphism of Hom-coassociative coalgebras. Then for any $c\in C$, we have

$\Delta_D\circ g(c)=(g\otimes g)\circ\Delta_C(c)=\sum(g\otimes g)(c_1\otimes c_2)=\sum g(c_1)\otimes g(c_2)\in \mathrm{Img}\otimes\mathrm{Img}$ and $U_D\circ g(c)=g\circ U_C(c)\in\mathrm{Img}$.

Therefore Img is a Sub-Hom-coassociative coalgebra.

Proposition 2.2 Suppose $\{V_i\}$ is a collection of Sub-Hom-coassociative coalgebras of V where V is a Hom-coassociative coalgebra, then $\sum V_i$ is also a Sub-Hom-coassociative coalgebra of V.

Proof Obviously, $\sum V_i$ is a subspace of V. We have

$\Delta(\sum V_i)=\sum\Delta(V_i)\subseteq\sum(V_i\otimes V_i)\subseteq\sum V_i\otimes\sum V_i$ and $U(\sum V_i)=\sum U(V_i)\subseteq\sum V_i$.

So $\sum V_i$ is also a Sub-Hom-coassociative coalgebra of V.

Proposition 2.3 Assume V is a Hom-coassociative coalgebra and V_1 is a Sub-Hom-coassociative coalgebra of V, then the inclusion map i: $V_1\rightarrow V$ must be a homomorphism of Hom-coassociative coalgebras and $\mathrm{Ker}i^*$ is exactly $V_1^{\perp}$.

Proof It′s easy to check that the inclusion map i: $V_1\rightarrow V$ is a homomorphism of Hom-coassociative coalgebras.

For any $v_1^*\in V_1^{\perp}$, $v_1\in V_1$, $\langle i^*(v_1^*),v_1\rangle=\langle v_1^*,i(v_1)\rangle=\langle v_1^*,v_1\rangle=0$, So $i^*(v_1^*)=0$, i.e. $v_1^*\in\mathrm{Ker}i^*$, thus $V_1^{\perp}\subseteq\mathrm{Ker}i^*$.

For any $v^*\in\mathrm{Ker}i^*$, $v_1\in V_1$, $\langle i^*(v^*),v_1\rangle=\langle v^*,i(v_1)\rangle=\langle v^*,v_1\rangle=0$, So $v^*\in V_1^{\perp}$ i.e. $\mathrm{Ker}i^*\subseteq V_1^{\perp}$. Therefore $\mathrm{Ker}i^*=V_1^{\perp}$.

Theorem 2.1 Let V be a Hom-coassociative coalgebra, then

(Ⅰ) If $V_1\subseteq V$ is a Sub-Hom-coassociative coalgebra, then $V_1^{\perp}\subseteq V^*$ is an ideal of V^*.

(Ⅱ) Suppose $I\subseteq V^*$ is an ideal, then $I^{\perp}\subseteq V$ is a Sub-Hom-coassociative coalgebra of V.

(Ⅲ) Suppose J is a subspace of V, then J is a Sub-Hom-coassociative coalgebra of V if and only if $J^{\perp}\subseteq V^*$ is an ideal of V^*.

Proof (Ⅰ) Following Proposition 2.3, making the inclusion map i: $V_1\rightarrow V$, then $V_1^{\perp}=\mathrm{Ker}i^*$. Clearly, $V_1^{\perp}\subseteq V^*$ is an ideal of V^*.

(Ⅱ) First, we verify $\Delta(x)\in I^{\perp}\otimes I^{\perp}$ for any $x\in I^{\perp}$ by using the method of contradiction. Let $\Delta(x)=\sum_{i=1}^{m}x_i\otimes y_i$, suppose $x_1\notin I^{\perp}$, then there is $e\in I$ s.t. $\langle e,x_1\rangle\neq 0$. Without loss of generality, assume $\{y_i\}$ is linearly independent and $\{y_i^*\}$ is the dual system of $\{y_i\}$, i.e.

$$\langle y_i^*,y_j\rangle=\begin{cases}1, i=j,\\0, i\neq j.\end{cases}$$

Let $y_1^*=f\in V^*$, since $I\subseteq V^*$ is an ideal, then $ef\in I$, $0=\langle ef,x\rangle=\langle\Delta^*(e\otimes f),x\rangle=\langle e\otimes f,\Delta(x)\rangle=\sum\langle e\otimes f,x_i\otimes y_i\rangle=\sum\langle e,x_i\rangle\langle f,y_i\rangle=\langle e,x_1\rangle\langle y_1^*,y_1\rangle=\langle e,x_1\rangle$,

a contradiction. Therefore $\Delta(x)\in I^{\perp}\otimes I^{\perp}$, i.e. $\Delta(I^{\perp})\subseteq I^{\perp}\otimes I^{\perp}$.

For any $x\in I^{\perp}$, $e\in I\subseteq V^*$, $\langle e,U(x)\rangle=\langle U^*e,x\rangle=U^*e(x)=0$, So $U(x)\in I^{\perp}$ i.e. $U(I^{\perp})\subseteq I^{\perp}$. Therefore $I^{\perp}\subseteq V$ is a Sub-Hom-coassociative coalgebra of V.

(Ⅲ) According to the results of (Ⅰ) and (Ⅱ), we can easily get (Ⅲ).

Theorem 2.2 Assume (V,Δ,U,X) is a Hom-coassociative coalgebra, (V^*,Δ^*,U^*,X^*) is the dual Hom-associative algebra of V, W is a Sub-Hom-coassociative coalgebra of V. If $U^*(W^{\perp})\subseteq W^{\perp}$, then $V^*/W^{\perp}\cong W^*$ is Hom-associative algebras.

Proof By Theorem 2.1, $W^{\perp}$ is an ideal of V^*. Making the inclusion map i: $W\rightarrow V$, According to Proposition 1.1 and Proposition 2.3, $V^*/W^{\perp}$ is a Hom-associative algebra and $\mathrm{Ker}i^*=W^{\perp}=\mathrm{Ker}\varsigma$ where ς: $V^*\rightarrow V^*/W^{\perp}$ is a natural map. Therefore $V^*/W^{\perp}\cong W^*$ is Hom-associative algebras.

Proposition 2.4[3] Suppose V is a vector space with subspaces $\{V_i\}$, U, W. Let $\{X_i\}$ be subspaces of V^*. Then

(i) $\bigcap V_i^{\perp}=(\sum V_i)^{\perp}$,

(ii) $U^{\perp}+W^{\perp}=(U\bigcap W)^{\perp}$,

(iii) $\bigcap X_i^{\perp}=(\sum X_i)^{\perp}$.

Theorem 2.3 The intersection of Sub-Hom-coassociative coalgebras is again a Sub-Hom-coassociative coalgebra.

Proof Suppose $\{V_i\}$ is a collection of Sub-Hom-coassociative coalgebras of a Hom-coassociative coalgebra V. By Theorem 2.1(Ⅰ) we know $V_i^{\perp}$ is an ideal of V^*, so $\sum V_i^{\perp}$ is also an ideal. By Theorem 2.1(Ⅱ), we get $(\sum V_i^{\perp})^{\perp}$ is a Sub-Hom-coassociative coalgebra of V. But according to Proposition 2.4, we have $(\sum V_i^{\perp})^{\perp}=\bigcap V_i^{\perp\perp}=\bigcap V_i$, therefore $\bigcap V_i$ is again a Sub-Hom-coassociative coalgebra.

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