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加权 Hardy空间上的有界复合算子的伴随表达式

Representation of Adjoint of Composition Operator on Weighted Hardy Space

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摘要: 给出加权 Hardy空间上的有界复合算子的伴随表达式,并验证文献[3]中复合算子的 Cowen伴随表示定理为该伴随表达式的特例.

关键词: 复合算子 伴随表示 加权 Hardy空间
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Abstract Representation formula of adjoint of bounded composition operator on weighted Hardy spaceis are given, and we verify that Cowen′s adjoint representation theorem of composition operator in reference[3] is special case of our representation formula.

Key words composition operator, representation of adjoint, weighted Hardy space

在复合算子理论中,求出复合算子的伴随表达是一个很有趣的问题[1,2],围绕它可以开展许多研究工作[3-5],但是对一般的复合算子,到现在为止还没有得出关于它的明确表达式.本文给出加权 Hardy空间 $H^2(U)$ 上的有界复合算子伴随的表达式并验证了复合算子中的 Cowen伴随表示定理为其特例.

1 相关概念

设 D 为复平面 C 上的以零点为圆心的单位圆盘, $H(D)$ 为 D 上的所有解析函数组成的空间.令 $f(z), g(z) \in H(D), f(z) = \sum_{n=0}^{\infty} f_n z^n, g(z) = \sum_{n=0}^{\infty} g_n z^n$; 令 $\|f\|_U^2 = \sum_{n=0}^{\infty} |f_n|^2 U^2(n)$, $\langle f, g\rangle_U = \sum_{n=0}^{\infty} f_n \overline{g_n} U^2(n)$, 其中 $U(n) > 0 (n \geq 0)$; 设 $H^2(U) = \{f \in H(D): \|f\|_U < +\infty\}$.易证 $\|\cdot\|_U$ 为 $H^2(U)$ 的范数且使得 $H^2(U)$ 成为 Hilbert空间,本文称 $H^2(U)$ 为加权 Hardy空间.当 $U(n) \equiv 1 (n \geq 0)$ 时, $H^2(U)$ 为 H^2,即经典 Hardy空间.

设解析映射 $\varphi: D \to D$,复合算子 C_φ 定义为

$C_\varphi f = f \circ \varphi, \forall f \in H(D)$.

由著名的 Littlewood 从属原理[6] 可推出:任一 D 上的解析自映射 φ 所诱导的线性算子 C_φ 在 H^2 上有界.为方便起见, $\|\cdot\|_\beta$ 简记为 $\|\cdot\|_2$, $\langle \cdot, \cdot \rangle_\beta$ 简记为 $\langle \cdot, \cdot \rangle$.

2 主要结论

引理 2.1 设 H 为可分 Hilbert 空间, $A = \{\zeta_n, n \geq 1\}$ 为其规范正交基, T 为 H 上的有界线性算子, $M = (\langle T\zeta_p, \zeta_q \rangle)(p, q \geq 1)$,此处 M 是 p 行 q 列元素为 $\langle T\zeta_p, \zeta_q \rangle$ 的无穷阵.若 $\forall f \in H, f = \sum_{n=1}^{\infty} f_n \zeta_n, \tilde{f} = (f_1, f_2, \cdots, f_n, \cdots)$,则 $\widetilde{Tf} = \tilde{f}M, \widetilde{T^*f} = \tilde{f}\overline{M}^T$.

证明 设 $g = \sum_{n=1}^{\infty} g_n \zeta_n \in H, \bar{\tilde{g}} = (\bar{g}_1, \bar{g}_2, \cdots, \bar{g}_n, \cdots)$,则 $\widetilde{Tf}\,\bar{\tilde{g}}^T = \langle Tf, g \rangle = \langle T(\sum_{p=1}^{\infty} f_p \zeta_p), \sum_{q=1}^{\infty} g_q \zeta_q \rangle = \sum_{p=1}^{\infty}\sum_{p=1}^{\infty} f_p \bar{g}_q \langle T\zeta_p, \zeta_q \rangle = \tilde{f}M\,\bar{\tilde{g}}^T$,由 g 的任意性,有 $\widetilde{Tf} = \tilde{f}M$.同样, $\widetilde{T^*f}\,\bar{\tilde{g}}^T = \langle T^*f, g \rangle = \langle f, Tg \rangle = \langle \sum_{q=1}^{\infty} f_q \zeta_q, T(\sum_{p=1}^{\infty} g_p \zeta_p) \rangle = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} f_q \bar{g}_p \langle \zeta_q, T\zeta_p \rangle = \tilde{f}\overline{M}^T\bar{\tilde{g}}^T$,再由 g 的任意性有 $\widetilde{T^*f} = \tilde{f}\overline{M}^T$.

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定理 2.1 若 $\varphi \in H(D)$ 将 D 映入 D，C_φ 为 $H^2(\beta)$ 上的有界线性算子，$f \in H^2(\beta)$，$\varphi(z) = \sum_{n=0}^{\infty}\varphi_n z^n$，$f(z) = \sum_{n=0}^{\infty} f_n z^n$，则

$$C_\varphi^* f(z) = \sum_{n=0}^{\infty}\frac{z^n}{\beta^2(n)}\sum_{q=0}^{\infty} f_q\beta^2(q)\cdot \sum_{s_1+\cdots+s_n=q,s_1,\cdots,s_n\geqslant 0}\bar{\varphi}_{s_1}\cdots\bar{\varphi}_{s_n}.$$

证明 设 $\tau(z)=z$，由于 $\{\frac{\tau^n}{\beta(n)}, n\geqslant 0\}$ 为 $H^2(\beta)$ 的规范正交基，则

$$m_{p,q} := \langle C_\varphi\frac{\tau^p}{\beta(p)},\frac{\tau^q}{\beta(q)}\rangle_\beta = \frac{1}{\beta(p)\beta(q)}\langle\varphi^p,\tau^q\rangle_\beta = \frac{\beta(q)}{\beta(p)}\sum_{s_1+\cdots+s_p=q,s_1,\cdots,s_p\geqslant 0}\varphi_{s_1}\cdots\varphi_{s_p}.$$

设 $f(z) = \sum_{n=0}^{\infty} f_n z^n = \sum_{n=0}^{\infty}\beta(n) f_n\frac{z^n}{\beta(n)}$，$\tilde{f} = (\beta(0)f_0,\cdots,\beta(n)f_n,\cdots)$，$M = (m_{p,q})$，由引理 2.1，有

$$(F_0,\cdots,F_n,\cdots) := \widetilde{C_\varphi^* f} = \tilde{f}\,\overline{M}^T.$$ 故

$$F_n = \sum_{q=0}^{\infty}\overline{m}_{n,q} f_q\beta(q) = \sum_{q=0}^{\infty}\frac{\beta^2(q)}{\beta(n)} f_q\sum_{s_1+\cdots+s_n=q,s_1,\cdots,s_n\geqslant 0}\bar{\varphi}_{s_1}\cdots\bar{\varphi}_{s_n},$$

$$C_\varphi^* f(z) = \sum_{n=0}^{\infty} F_n\frac{z^n}{\beta(n)} = \sum_{n=0}^{\infty}\frac{z^n}{\beta^2(n)}\sum_{q=0}^{\infty} f_q\beta^2(q)\sum_{s_1+\cdots+s_n=q,s_1,\cdots,s_n\geqslant 0}\bar{\varphi}_{s_1}\cdots\bar{\varphi}_{s_n}.$$

注 文献 [3]的定理 2为定理 2.1的特例

推论 2.1 若 h$(z) = (\text{T}z+\text{U})/(\text{V}z+\text{W})$ 为 D 上的线性分式自映射，此处 TW $-$ UV $\neq 0$，e$(z) = \frac{\bar{\text{T}}z - \bar{\text{V}}}{-\bar{\text{U}}z+\bar{\text{W}}}$，$g(z) = \frac{1}{-\bar{\text{U}}z+\bar{\text{W}}}$，$h(z) = \text{V}z+\text{W}$，则在 H^2 上 $C_\text{h}^* = T_g C_\text{e} T_h^*$.

证明 由定理 2.1并注意 U$(n) \equiv 1 (n\in \text{N}\cup\{0\})$，有

$$C_\text{h}^* f(z) = \sum_{n=0}^{\infty} f_0\bar{\text{h}}_0^n z^n + \sum_{n=1}^{\infty} z^n\sum_{q=1}^{\infty} f_q\sum_{p=0}^{n-1} C_n^{n-p-1}\cdot \bar{\text{h}}_0^{n-p-1}\sum_{s_1+\cdots+s_{p+1}=q,s_1,\cdots,s_{p+1}>0}\bar{\text{h}}_{s_1}\cdots\bar{\text{h}}_{s_{p+1}} = \frac{f_0}{1-\bar{\text{h}}_0 z} +$$
$$\sum_{q=1}^{\infty} f_q\sum_{n=1}^{\infty}\sum_{p=0}^{n-1} z^n C_n^{n-p-1}\bar{\text{h}}_0^{n-p-1}\cdot \sum_{s_1+\cdots+s_{p+1}=q,s_1,\cdots,s_{p+1}>0}\bar{\text{h}}_{s_1}\cdots\bar{\text{h}}_{s_{p+1}} = \frac{f_0}{1-\bar{\text{h}}_0 z} +$$
$$\sum_{q=1}^{\infty} f_q\sum_{p=0}^{\infty}\sum_{n=p+1}^{\infty} z^n C_n^{n-p-1}\bar{\text{h}}_0^{n-p-1}\sum_{s_1+\cdots+s_{p+1}=q,s_1,\cdots,s_{p+1}>0}\bar{\text{h}}_{s_1}\cdots\bar{\text{h}}_{s_{p+1}} = \frac{f_0}{1-\bar{\text{h}}_0 z} + \sum_{q=1}^{\infty} f_q\sum_{p=0}^{q-1}\frac{z^{p+1}}{(1-\bar{\text{h}}_0 z)^{p+2}}\cdot$$
$$\sum_{s_1+\cdots+s_{p+1}=q,s_1,\cdots,s_{p+1}>0}\bar{\text{h}}_{s_1}\cdots\bar{\text{h}}_{s_{p+1}}.$$

再分 2步来证明.

(1) 若 $\text{h}(z) = \frac{b+cz}{1-az} = (b+cz)\sum_{n=0}^{\infty} a^n z^n = \sum_{n=0}^{\infty} a^n b z^n + \sum_{n=0}^{\infty} a^n c z^{n+1} = b + \sum_{n=1}^{\infty}(a^n b + a^{n-1}c)z^n = b + \sum_{n=1}^{\infty} a^{n-1}(ab+c)z^n.$

所以有

$$C_\text{h}^* f(z) = \frac{\hat{f}_0}{1-\bar{b}z} + \sum_{n=1}^{\infty}\hat{f}_n\sum_{p=0}^{n-1}\sum_{s_1+\cdots+s_{p+1}=n,s_t>0}\frac{z^{p+1}\prod_{t=1}^{p+1}\overline{(ab+c)}\,\bar{a}^{s_t-1}}{(1-\bar{b}z)^{p+2}} = \frac{\hat{f}_0}{1-\bar{b}z} +$$
$$\sum_{n=1}^{\infty}\hat{f}_n\sum_{p=0}^{n-1}\sum_{s_1+\cdots+s_{p+1}=n,s_t>0}\frac{(\bar{a}\bar{b}+\bar{c})^{p+1}\bar{a}^{n-p-1}z^{p+1}}{(1-\bar{b}z)^{p+2}} = \frac{f_0}{1-\bar{b}z} + \sum_{n=1}^{\infty}\frac{\hat{f}_n z(\bar{a}\bar{b}+\bar{c})}{(1-\bar{b}z)^2}\sum_{p=0}^{n-1} C_{n-1}^p\bar{a}^{n-1-p}$$
$$[\frac{(\bar{a}\bar{b}+\bar{c})z}{1-\bar{b}z}]^p = \frac{\hat{f}_0}{1-\bar{b}z} + \sum_{n=1}^{\infty}\frac{\hat{f}_n z(\bar{a}\bar{b}+\bar{c})}{(1-\bar{b}z)^2}[\bar{a} + \frac{(\bar{a}\bar{b}+\bar{c})z}{1-\bar{b}z}]^{n-1} = \frac{\hat{f}_0}{1-\bar{b}z} + \sum_{n=1}^{\infty}\frac{\hat{f}_n z(\bar{a}\bar{b}+\bar{c})}{(1-\bar{b}z)^2}(\frac{\bar{a}+\bar{c}z}{1-\bar{b}z})^{n-1}.$$

若 $\text{h}(z) = \frac{a'z+b'}{c'z+d'} = \frac{\frac{b'}{d'}+\frac{a'}{d'}z}{1-(-\frac{c'}{d'})z}$，$(d'\neq 0)$

设 $a = -\frac{c'}{d'}$，$b = \frac{b'}{d'}$，$c = \frac{a'}{d'}$，由此，

$$C_\text{h}^* f(z) = \frac{\hat{f}_0}{1-\frac{\bar{b}'}{\bar{d}'}z} + \sum_{n=1}^{\infty}\frac{\hat{f}_n z(-\frac{\bar{c}'}{\bar{d}'}\frac{\bar{b}'}{\bar{d}'}+\frac{\bar{a}'}{\bar{d}'})}{(1-\frac{\bar{b}'}{\bar{d}'}z)^2}(\frac{-\frac{\bar{c}'}{\bar{d}'}+\frac{\bar{a}'}{\bar{d}'}z}{1-\frac{\bar{b}'}{\bar{d}'}z})^{n-1} =$$
$$\frac{\hat{f}_0\bar{d}'}{\bar{d}'-\bar{b}'z} + \sum_{n=1}^{\infty}\frac{\hat{f}_n z(-\bar{c}'\bar{b}'+\bar{a}'\bar{d}')}{(\bar{d}'-\bar{b}'z)^2}(\frac{-\bar{c}'+\bar{a}'z}{\bar{d}'-\bar{b}'z})^{n-1}.$$

(2) 设 $f(z) = \sum_{n=0}^{\infty}\hat{f}_n z^n$，$g(z) = \sum_{n=0}^{\infty}\hat{g}_n z^n$，$h(z) = \sum_{n=0}^{\infty}\hat{h}_n z^n$，$T_h^* f = \sum_{n=0}^{\infty} a_n z^n$，则

$$h(z)g(z) = \sum_{n=0}^{\infty}(\sum_{k=0}^{n}\hat{h}_k\hat{g}_{n-k})z^n, \sum_{n=0}^{\infty} a_n\bar{\hat{g}}_n = \langle T_h^* f(z), g(z)\rangle =$$

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一个 Lucas三角形,则$\frac{L_{n+k}}{d}\equiv 1(\mathrm{mod}4)$,其中 $d=(L_n, L_{n+k})$.

证明　由引理 3知,$d=(L_n, L_{n+k})=(L_{n+k}, \frac{1}{2}L_n)$.设此三角形底边上的高为 h,则 $h^2=L_{n+k}^2-\frac{1}{4}L_n^2$,变形得$\frac{h^2}{d^2}=\frac{L_{n+k}^2}{d^2}-\frac{L_n^2}{4d^2}$即$(\frac{h}{d})^2+(\frac{L_n}{2d})^2=(\frac{L_{n+k}}{d})^2$,$(\frac{h}{d},\frac{L_n}{2d},\frac{L_{n+k}}{d})$为本原商高数,于是$\frac{L_{n+k}}{d}\equiv 1(\mathrm{mod}4)$.

引理 5　如果边长为 L_n, L_{n+k}, L_{n+k}的三角形是一个 Lucas三角形,则$\frac{L_{n+k}}{d}+\frac{L_n}{2d}$,$\frac{L_{n+k}}{d}-\frac{L_n}{2d}$都是平方数,其中 $d=(\frac{1}{2}L_n, L_{n+k})$.

证明　由引理 4知$\frac{h^2}{d^2}=\frac{L_{n+k}^2}{d^2}-\frac{L_n^2}{4d^2}=(\frac{L_{n+k}}{d}+\frac{L_n}{2d})(\frac{L_{n+k}}{d}-\frac{L_n}{2d})$,而$\frac{h^2}{d^2}$是一个平方数,并且$\frac{L_{n+k}}{d}+\frac{L_n}{2d}$和$\frac{L_{n+k}}{d}-\frac{L_n}{2d}$互素,故$\frac{L_{n+k}}{d}+\frac{L_n}{2d}$,$\frac{L_{n+k}}{d}-\frac{L_n}{2d}$都是平方数.

2　主要结论

定理 1　不存在边长为 L_{n-k}, L_n, L_n ($1\leqslant k<n$)的 Lucas三角形.

证明　假设存在以 L_n, L_{n+k}, L_{n+k}为边长的 Lucas三角形.由引理 4知$\frac{L_{n+k}}{d}\equiv 1(\mathrm{mod}4)$,则$\frac{L_{n+k}}{d}+\frac{L_n}{2d}\equiv 1+\frac{L_n}{2d}(\mathrm{mod}4)$,$\frac{L_{n+k}}{d}-\frac{L_n}{2d}\equiv 1-\frac{L_n}{2d}(\mathrm{mod}4)$.若$\frac{L_n}{2d}\equiv 1(\mathrm{mod}4)$,则$\frac{L_{n+k}}{d}+\frac{L_n}{2d}\equiv 2(\mathrm{mod}4)$,由引理 5知这不可能成立;若$\frac{L_n}{2d}\equiv 2(\mathrm{mod}4)$,则$\frac{L_{n+k}}{d}+\frac{L_n}{2d}\equiv 3(\mathrm{mod}4)$,由引理 5知这也不可能成立;若$\frac{L_n}{2d}\equiv 3(\mathrm{mod}4)$,则$\frac{L_{n+k}}{d}+\frac{L_n}{2d}\equiv 2(\mathrm{mod}4)$,同样由引理 5知这不可能成立.故$\frac{L_n}{2d}\equiv 1(\mathrm{mod}4)$,于是 $8|L_n$.

对 Lucas数列$\{L_n\}$取模 8,得到剩余类周期为 12的剩余类序列: 1, 3, 4, 7, 3, 2, 5, 7, 4, 3, 7, 2,…… 所以 $L_n\neq 0(\mathrm{mod}8)$,这与 $8|L_n$矛盾.故假设不成立,即不存在以 L_n, L_{n+k}, L_{n+k}为边长的 Lucas三角形.

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$\langle f(z), T_h g(z)\rangle=\langle f(z), h(z)g(z)\rangle=\sum_{n=0}^{\infty}\hat f_n(\sum_{k=0}^{n}\overline{\hat g_k}\,\overline{\hat h_{n-k}})=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\hat f_n\overline{\hat g_k}\,\overline{\hat h_{n-k}}=\sum_{k=0}^{\infty}\overline{\hat g_k}\sum_{n=k}^{\infty}\hat f_n\overline{\hat h_{n-k}}=\sum_{n=0}^{\infty}\overline{\hat g_n}\sum_{k=n}^{\infty}\hat f_k\overline{\hat h_{k-n}}$,

比较得 $a_n=\sum_{k=n}^{\infty}\hat f_k\overline{\hat h_{k-n}}$.

若 $h(z)=cz+d$,$\varphi(z)=\frac{\bar a z-\bar c}{-\bar b z+\bar d}$,$g(z)=\frac{1}{-\bar b z+\bar d}$,则 $a_n=\hat f_n\bar d+\hat f_{n+1}\bar c$,

$T_g C_\varphi T_h^* f=T_g C_\varphi[\sum_{n=0}^{\infty}(\hat f_n\bar d+\hat f_{n+1}\bar c)z^n]=\frac{1}{-\bar b z+\bar d}\sum_{n=0}^{\infty}(\hat f_n\bar d+\hat f_{n+1}\bar c)[\frac{\bar a z-\bar c}{-\bar b z+\bar d}]^n=\frac{\hat f_0\bar d}{-\bar b z+\bar d}+\frac{1}{-\bar b z+\bar d}\sum_{n=1}^{\infty}\hat f_n\{\bar d[\frac{\bar a z-\bar c}{-\bar b z+\bar d}]^n+\bar c[\frac{\bar a z-\bar c}{-\bar b z+\bar d}]^{n-1}\}=\frac{\hat f_0\bar d}{-\bar b z+\bar d}+\frac{1}{-\bar b z+\bar d}\cdot\sum_{n=1}^{\infty}\hat f_n\frac{z(-\bar c\bar b+\bar a\bar d)}{\bar d-\bar b z}[\frac{\bar a z-\bar c}{-\bar b z+\bar d}]^{n-1}$.

比较可以得出推论 2.1成立.

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