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On the Uniqueness of Generalized Eigenmatrices*
广义特征矩阵的唯一性

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Abstract: The general form of the maximal Jordan chains of the defective matrix is getten with the notation of depths of the generalized eigenvectors. For two invertible matrices P and S such that $PJP^{-1} = SJS^{-1}$, we find that there exists a block matrix H with upper triangular Toeplitz block matrices laying on its principal block diagonal such that $S = PH$, which allows to prove the uniqueness of generalized eigenmatrices.

Key words: matrix, generalized eigenmatrix, Jordan canonical form, Jordan chain, Toeplitz block submatrix

摘要:利用广义特征向量的深度,获得极大若当链的一般形式,并推导出在满足 $PJP^{-1}= SJS^{-1}$的 2个可逆矩阵 P和 S之间存在一个主对角线上具有上三角分块 Toeplitz子阵的可逆矩阵 H,使得 $S= PH$,从而证明广义特征矩阵的唯一性.

关键词:矩阵 广义特征矩阵 若当标准型 若当链 Toeplitz子阵
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Jordan canonical forms and Jordan chains of defective matrices are very important in both pure and applied mathematics. There are a number of publications about them recently, especially on the research of M-matrix[1]. One of their classical applications is to express the fundamental solutions of a system of differential equations with constant coefficients[2]. But the Jordan chains are difficult to be obtained. The new concept of generalized eigenmatrices was given in reference[3~ 6] to enhance the computation efficiency of matrix functions and their integrals, which enables us to avoid the computation of Jordan chains. The existence of the generalized eigenmatrices has been proved with a linear system whose coefficient matrix is a generalized Vandermonde matrix. In this paper we prove its uniqueness with the notation of depths of the generalized eigenvectors[1,7].

1 Basic concepts

In this paper, we always assume that A is a defective complex matrix of order n with the distinct eigenvalues $\lambda_1, \cdots, \lambda_d$. We denote the block diagonal matrix $\begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$[2] by $A_1 \oplus A_2$. With this notation, the Jordan canonical form of A is $J = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_d}(\lambda_d)$ with $J_{n_i}(\lambda_i) = J_{n_i(1)}(\lambda_i) \oplus \cdots \oplus J_{n_i(t(i))}(\lambda_i)$, where $J_{n_i(j)}(\lambda_i)$ is a Jordan block of order $n_i(j)$ with diagonal elements λ_i, super-diagonal elements 1, and the others 0. $\sum_{j=1}^{t(i)} n_i(j) = n_i$ is the algebraic multiplicity of λ_i. The $t(i)$ Jordan blocks corresponding to λ_i in J are presented in decreasing (nonincreasing) order with the largest block first, then the next largest, and so

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on, i.e., $n_i(1) \geqslant \cdots \geqslant n_i(j) \geqslant \cdots \geqslant n_i(t(i))$.

Let $N_{n_i(j)}$ be the nilpotent matrix obtained from $J_{n_i(j)}(\lambda_i)$ by replacing the diagonal λ_i by zeros, then

$$N^k_{n_i(j)} = \begin{bmatrix} 0 & \cdots & 1 & & \\ & \ddots & & \ddots & \\ & & 0 & \cdots & 1 \\ & & & \ddots & \vdots \\ & & & & 0 \end{bmatrix}, k = 2, \cdots, n_i(j) - 1.$$

Define $N^0_{n_i(j)}$ to be the identity matrix of order $n_i(j)$. Denote that $N_{n_i} = N_{n_i(1)} \oplus N_{n_i(2)} \oplus \cdots \oplus N_{n_i(t(i))}$, and let $N_i = O \oplus \cdots \oplus N_{n_i} \oplus \cdots \oplus O$ be a block matrix with the same partition as J, in which all blocks but the i-th are non-zero.

Definition 1[3] Let P be an invertible matrix such that $A = PJP^{-1}$. We call $A_i^{(k)} = PN_i^k P^{-1}$ the generalized eigenmatrix of A corresponding to λ_i, where $k = 0, 1, \cdots, n_i - 1$.

We denote by $\mathrm{null}(A - \lambda_i I)^{n_i(1)}$ the generalized eigenspace corresponding to λ_i, that is,

$$\mathrm{null}(A - \lambda_i I)^{n_i(1)} = \{\xi | (A - \lambda_i I)^{n_i(1)} \xi = O\}.$$

Let $\xi_1, \xi_2, \cdots, \xi_m$ be a Jordan chain corresponding to λ_i, that is,

$$A\xi_1 = \lambda_i \xi_1, A\xi_2 = \lambda_i \xi_2 + \xi_1, \cdots, A\xi_m = \lambda_i \xi_m + \xi_{m-1}.$$

The leading vector ξ_1 is an eigenvector, and the others are generalized eigenvectors[1,7,8]. If the linear system $(A - \lambda_i I)X = \xi_m$ is inconsistent, then the Jordan chain $\xi_1, \xi_2, \cdots, \xi_m$ is called maximal. For a maximal Jordan chain $\xi_1, \xi_2, \cdots, \xi_m$, the depths of vectors $\xi_1, \xi_2, \cdots, \xi_m$ are defined as $m - 1, m - 2, \cdots, 0$ respectively. We denote by $dp(\xi_k)$ the depth of ξ_k[1,7]. For a given vector $\xi \in \mathrm{null}(A - \lambda_i I)^{n_i(1)}$, $dp(\xi) = j$ if and only if the linear system $(A - \lambda_i I)^{j-1} X = \xi$ is consistent, while $(A - \lambda_i I)^j X = \xi$ is inconsistent.

Suppose that

$$P = (P_1, P_2, \cdots, P_d), P_i = (P_{i,1}, P_{i,2}, \cdots, P_{i,t(i)}),$$
$$P_{i,j} = (\xi_1(i,j), \xi_2(i,j), \cdots, \xi_{n_i(j)}(i,j)),$$

where $j = 1, \cdots, t(i)$, and $\xi_1(i,j), \xi_2(i,j), \cdots, \xi_{n_i(j)}(i,j)$ is a maximal Jordan chain corresponding to $J_{n_i(j)}(\lambda_i)$. The length of this chain is just equal to $n_i(j)$, the order of $J_{n_i(j)}(\lambda_i)$, and the depth of the eigenvector $\xi_1(i,j)$ is equal to $n_i(j) - 1$. When j runs from 1 to $t(i)$, $\xi_1(i,j), \xi_2(i,j), \cdots, \xi_{n_i(j)}(i,j)$ form a Jordan base of $\mathrm{null}(A - \lambda_i I)^{n_i(1)}$. Thus P_i is an $n \times n_i$ matrix constituted by a Jordan basis of $\mathrm{null}(A - \lambda_i I)^{n_i(1)}$.

Denote the total number of Jordan blocks $J_{n_i(j)}(\lambda_i)$ in $J_{n_i}(\lambda_i)$ of all sizes $n_i(j) \geqslant m$ by $\omega(m)$. That is, if $n_i(1) \geqslant \cdots \geqslant n_i(a) \geqslant m > n_i(a+1)$, then $\omega(m) = a$.

2 Main results

Lemma 1[1,7] For complex numbers $a_1, a_2, \cdots, a_m$, and vectors $\xi_1, \xi_2, \cdots, \xi_m \in \mathrm{null}(A - \lambda_i I)^{n_i(1)}$, we have

$$dp(\sum_{j=1}^{m} a_j \xi_j) \geqslant \min\{dp(\xi_j): j = 1, 2, \cdots, m\}, \quad (1)$$

where $\sum_{j=1}^{m} a_j \xi_j \neq O$, and for $\xi_u, \xi_v \in \mathrm{null}(A - \lambda_i I)^{n_i(1)}$, we have the strict inequality

$$dp(\xi_u) \neq dp(\xi_v) \Rightarrow dp(\xi_u + \xi_v) = \min\{dp(\xi_u), dp(\xi_v)\}. \quad (2)$$

Proposition 1 Let $\alpha_1(i,j), \alpha_2(i,j), \cdots, \alpha_{n_i(j)}(i,j)$ be the general form of a maximal Jordan chain with length $n_i(j)$ corresponding to λ_i. Then

$$\alpha_k(i,j) = \sum_{h=1}^{\omega(n_i(j)-k+1)} \sum_{l=1}^{k} b_l^{(h,j)} \xi_{k+1-l}(i,h), k = 1, \cdots, n_i(j), \quad (3)$$

where $b_l^{(h,j)}$ are complex numbers, and

$$\sum_{h=\omega(n_i(j)+1)+1}^{\omega(n_i(j))} b_1^{(h,j)} \xi_1(i,h) \neq O.$$

Proof Since $\xi_1(i,h), \cdots, \xi_{k+1-l}(i,h), \cdots, \xi_{n_i(h)}(i,h)$ are the maximal Jordan chain corresponding to $J_{n_i(h)}(\lambda_i)$, we have

$$dp(\xi_{k+1-l}(i,h)) = n_i(h) - (k + 1 - l) > n_i(j) - k.$$

Now we prove the Proposition 1 by induction on k. If $k = 1$, then $dp(\alpha_1(i,j)) = n_i(j) - 1$, which implies that $\alpha_1(i,j)$ is a linear combination of $\xi_1(i,1), \xi_1(i,2), \cdots, \xi_1(i, \omega(n_i(j)))$, i.e.,

$$\alpha_1(i,j) = \sum_{h=1}^{\omega(n_i(j))} b_1^{(h,j)} \xi_1(i,h). \quad (4)$$

It follows from formulae (1) that $\sum_{h=\omega(n_i(j)+1)+1}^{\omega(n_i(j))} b_1^{h,j} \xi_1(i,h) \neq O$, by the fact $dp(\alpha_1(i,j)) = n_i(j) - 1$.

Suppose that $\alpha_k(i,j) = \sum_{h=1}^{Q(n_i(j)-k+1)} \cdot \sum_{l=1}^{k} b_l^{(h,j)} \eta_{k+1-l}(i,h)$ holds. Since $(A - \lambda_i I)\eta_{k+2-l} = \eta_{k+1-l}$, we can easily verify that $\sum_{h=1}^{Q(n_i(j)-k+1)} \cdot \sum_{l=1}^{k} b_l^{(h,j)} \eta_{k+2-l}(i,h)$ is a special solution of the linear system $(A - \lambda_i I)X = \alpha_k(i,j)$. Because $\eta_1(i,h), h = 1,\cdots,t(i)$, is a fundamental solutions of $(A - \lambda_i I)X = O$, the general solution of $(A - \lambda_i I)X = \alpha_k(i,j)$ is

$$X = \sum_{h=1}^{t(i)} c_h \eta_1(i,h) + \sum_{h=1}^{Q(n_i(j)-k+1)} \sum_{l=1}^{k} b_l^{(h,j)} \eta_{k+2-l}(i,h),$$

where $c_h, h = 1,\cdots,t(i)$, are scalars. Since $(A - \lambda_i I)\alpha_{k+1}(i,j) = \alpha_k(i,j)$, we can write

$$\alpha_{k+1}(i,j) = \sum_{h=1}^{Q(n_i(j)-k)} b_{k+1}^{(h,1)} \eta_1(i,h) + \sum_{h=Q(n_i(j)-k)+1}^{t(i)} c'_h \eta_1(i,h) + \sum_{h=1}^{Q(n_i(j)-k+1)} \sum_{l=1}^{k} b_l^{(h,j)} \eta_{k+2-l}(i,h). \quad (5)$$

Since $dp(\alpha_{k+1}(i,j)) = n_i(j) - k - 1$ and $dp(\eta_1(i,h)) < n_i(j) - k - 1$ for $h = Q(n_i(j) - k) + 1,\cdots,t(i)$, we have $c'_h = 0$ by Lemma 1. Reducing formulae(5), we get the general form of $\alpha_{k+1}(i,j)$, that is

$$\alpha_{k+1}(i,j) = \sum_{h=1}^{Q(n_i(j)-k)} \sum_{l=1}^{k+1} b_l^{(h,j)} \eta_{k+2-l}(i,h).$$

The proof is complete.

Proposition 2 Let P and S be invertible matrices such that $A = PJP^{-1} = SJS^{-1}$. Then there exists an invertible matrix H such that $S = PH$ with $H = H_1 \oplus H_2 \oplus \cdots \oplus H_d$ is a block matrix with the same partition as J, where $H_i = (H_{h,j})_{t(i)\times t(i)}$ and $H_{h,j}$ is the h-th block in the j-th block column of H_i. If $n_i(j) = n_i(h)$, then $H_{h,j}$ is an upper triangular Toeplitz matrix, i.e.,

$$H_{h,j} = \begin{bmatrix} b_1^{(h,j)} & b_2^{(h,j)} & \cdots & \cdots & b_{n_i(h)}^{(h,j)} \\ & b_1^{(h,j)} & b_2^{(h,j)} & \cdots & \cdots \\ & & \ddots & \ddots & \vdots \\ & & & b_1^{(h,j)} & b_2^{(h,j)} \\ & & & & b_1^{(h,j)} \end{bmatrix}_{n_i(h)\times n_i(h)}$$

or $$H_{h,j} = \begin{cases} [O, H_{h,h}]_{n_i(h)\times n_i(j)}, & \text{if } n_i(j) > n_i(h), \\ \begin{bmatrix} O \\ H_{h,h} \end{bmatrix}_{n_i(h)\times n_i(j)}, & \text{if } n_i(j) < n_i(h). \end{cases}$$

Proof Let

$S = (S_1, S_2, \cdots, S_d)$, $S_i = (S_{i,1}, S_{i,2}, \cdots, S_{i,t(i)})$, $S_{i,j} = (\alpha_1(i,j), \alpha_2(i,j), \cdots, \alpha_{n_i(j)}(i,j))$, where $\alpha_1(i,j), \alpha_2(i,j), \cdots, \alpha_{n_i(j)}(i,j)$ is a maximal Jordan chain with length $n_i(j)$, see formulae(3) and formulae(4).

When j runs from 1 to $t(i)$, the eigenvectors $\alpha_1(i,1), \cdots, \alpha_1(i,j), \cdots, \alpha_1(i,t(i))$ are obtained by formulae (4). Assume that they are linearly independent, then the column vectors of $S_{i,j}$, i.e., $\alpha_k(i,j), j = 1, 2, \cdots, t(i); k = 1, 2, \cdots, n_i(j)$, are also linearly independent. Thus the column vectors of S_i form a Jordan basis of $\text{null}(A - \lambda_i I)^{n_i(1)}$. According to formulae(3) and formulae(4), we have

$S_{i,j} = (\alpha_1(i,j), \cdots, \alpha_{n_i(j)}(i,j)) = (P_{i,1}, \cdots, P_{i,t(i)})(H_{1,j}^{\mathrm{T}}, \cdots, H_{t(i),j}^{\mathrm{T}})^{\mathrm{T}} = P_i(H_{1,j}^{\mathrm{T}}, \cdots, H_{t(i),j}^{\mathrm{T}})^{\mathrm{T}}$,

where $(H_{1,j}^{\mathrm{T}}, \cdots, H_{t(i),j}^{\mathrm{T}})^{\mathrm{T}}$ is the j-th block column of H_i. Thus

$S_i = (S_{i,1}, \cdots, S_{i,t(i)}) = (P_i(H_{1,1}^{\mathrm{T}}, \cdots, H_{t(i),1}^{\mathrm{T}})^{\mathrm{T}}, \cdots, P_i(H_{1,t(i)}^{\mathrm{T}}, \cdots, H_{t(i),t(i)}^{\mathrm{T}})^{\mathrm{T}}) = P_i H_i$,

$S = (S_1, \cdots, S_d) = (P_1 H_1, \cdots, P_d H_d) = (P_1, \cdots, P_d)(H_1 \oplus \cdots \oplus H_d) = PH$.

Since S and P are invertible, H is also invertible. The proof is complete.

Theorem 1 With respect to the order of $\lambda_1, \cdots, \lambda_d$, the generalized eigenmatrices $A_i^{(k)}, i = 1, \cdots, d; k = 0, 1, \cdots, n_i - 1$, are independent of P such that $A = PJP^{-1}$.

Proof Since $N_{n_i(h)}^{K} H_{h,j} = H_{h,j} N_{n_i(j)}^{k}$ [2], we can easily verify that

$HN_i^k = (H_1 \oplus \cdots \oplus H_i \oplus \cdots \oplus H_d)(O \oplus \cdots \oplus N_{n_i}^k \oplus \cdots \oplus O) = H_i N_{n_i}^k = (H_{h,j})_{t(i)\times t(i)} (N_{n_i(1)}^k \oplus N_{n_i(2)}^k \oplus \cdots \oplus N_{n_i(t(i))}^k) = (H_{h,j} N_{n_i(j)}^k)_{t(i)\times t(i)} = (N_{n_i(h)}^k H_{h,j})_{t(i)\times t(i)} = N_i^k H$.

Now we compute the generalized eigenmatrices by replacing P with S in definition.

$SN_i^k S^{-1} = (PH)N_i^k(H^{-1}P^{-1}) = PN_i^k(HH^{-1})P^{-1} = PN_i^k P^{-1} = A_i^k$.

The proof is complete.

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$\frac{e_1}{t^2}$, hence the first equation of formulae(3) becomes

$$a^3 - db_1^3t^3 = kt^3, \quad (6)$$

From formulae(6), we see that $t|a$. Let $a = ta_1$, thus we obtain

$$a_1^3 - db_1^3 = k. \quad (7)$$

Note that $k|(d^3 - d)$, by lemma 1, the solution of formulae(7) satisfies $|a_1| < C_1, |b_1| < C_1$, where C_1 is an effectively computable constant depending upon d.

Since $a = ta_1, b = tb_1, e = te_1, e_1 = kt^2$, the second equation of formulae(3) gives

$$a_1 - db_1 = kt^2. \quad (8)$$

Therefore $|t| < \frac{(|a_1| + |db_1|)}{|k|}$, hence $|a| = |ta_1| < C, |b| = |tb_1| < C$, where C is an effectively computable constant depending upon d. This proves the theorem 1.

Theorem 2 The only integer solutions of the equation

$$y(y+1)(y+2) = 2x(x+1)(x+2) \quad (9)$$

are given by $(x,y) = (-2,-2), (-2,0), (-2,-1), (0,-1), (0,0), (0,-2), (-1,-1), (-5,-6)$ and $(3,4)$.

Proof Let $d = 2$, then $k|(d^3 - d) = 6$, formulae(7) and formulae(8) give

$$a_1^3 - 2b_1^3 = \pm 1, a_1 - 2b_1 = \pm t^2, \quad (10)$$

$$\text{or } a_1^3 - 2b_1^3 = \pm 2, a_1 - 2b_1 = \pm 2t^2, \quad (11)$$

$$\text{or } a_1^3 - 2b_1^3 = \pm 3, a_1 - 2b_1 = \pm 2t^2, \quad (12)$$

$$\text{or } a_1^3 - 2b_1^3 = \pm 6, a_1 - 2b_1 = \pm 6t^2, \quad (13)$$

The first equation of formulae(10) has only solutions $a_1 = \pm 1, b_1 = 0$ and $a_1 = \mp 1, b_1 = \mp 1$. These give $t^2 = 1, t = \pm 1$, further give $(y,x) = (-2,-1), (0,-1), (-2,-2), (0,0)$ respectively.

The first equation of formulae(11) gives $2|a_1$, let $a_1 = 2a_2$, hence we have $4a_2^3 - b_1^3 = \pm 1$. By lemma 2, it gives $a_1 = 0, b_1 = \mp 1$, therefore $(y,x) = (-1,-2), (-1,0)$ respectively.

The first equation of formulae(12) has only solutions $(a_1, b_1) = (\pm 1, \mp 1), (\mp 5, \mp 4)$, so $t^2 = 1$. These give solutions $(y,x) = (0,-2), (-2,0), (-6,-5), (4,3)$ respectively.

The first equation of formulae(13) becomes

$$4a_2^3 - b_1^3 = \pm 3, a_1 = 2a_2. \quad (14)$$

From lemma 2, formulae(14) has only solutions $a_2 = \pm 1, b_1 = \pm 1$, and hence $a_1 = \pm 2, b_1 = \pm 1$. Therefore the second equation of formulae(13) gives $\pm 6t^2 = a_1 - 2b_1 = 0$, so $a = ta_1 = 0, b = tb_1 = 0$, this gives $(x,y) = (-1,-1)$ by $a = y+1, b = x+1$. The proof is completed.

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