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丢番图方程 $y(y+1)(y+2)=2x(x+1)(x+2)$ 的整数解
The Integer Solution on the Diophantine Equation $y(y+1)(y+2)=2x(x+1)(x+2)$

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Abstract Let d be a given integer, all integer solutions of the diophantine equation $y(y+1)(y+2)=dx(x+1)(x+2)$ are effective determined, i.e. $\max\{|x|,|y|\}<C$ (effectively computable constant depending only d). In particular, all solutions can be determined when $d=2$.

Key words Diophantine equation, integer solutiion, effective constant

摘要: 证明丢番图方程 $y(y+1)(y+2)=dx(x+1)(x+2)$ 的所有整数解满足 $\max\{|x|,|y|\}<C$,其中,d 是一个给定整数,C 为仅依赖于 d 的有效常数.特别地,给出当 $d=2$ 时方程的全部整数解.

关键词: 丢番图方程 整数解 有效常数

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Let d be a given integer, we will show that all integer solutions of the diophantine equation $y(y+1)(y+2)=dx(x+1)(x+2)$ are effective determine, i.e. $\max\{|x|,|y|\}<C$ (effectively computable constant depending only d). In particular, all solutions can be determined when $d=2$.

1 Lemmas

Lemma 1[1] Let d,k are given integers, then the solutions of $x^3-dy^3=k$ satisfy the inequality $\max(|x|,|y|)<C$, where C is an effectively comutable constant depending only on d,k.

Lemma 2[2,3] Let a,b,c be positive integers, $a>b>1$, $c=1,3$, $(ab,c)=1$, $b=1$, if $c=3$, then the equation

$$ax^3+by^3=c \quad (1)$$

has at most one integer solution (x,y), and for this $c^{-1}(x\sqrt[3]{a}+y\sqrt[3]{b})^3$ is either the fundamental unit or its square in the field $Q(\sqrt[3]{d})$ defined by $Q(\sqrt[3]{ab^2})$, excluding however, the equation $2x^3+y^3=3$ which has the two solutions $(1,1)$ and $(4,-5)$.

2 Main results

Theorem 1 Let d be a given integer, then the solutions of

$$y(y+1)(y+2)=dx(x+1)(x+2) \quad (2)$$

satisfy the inequality $\max\{|x|,|y|\}<C$, where C is an effectively computable constant depending only d.

Proof Let $a=y+1$, $b=x+1$, then formulae (2) may be transformed into

$a^3-a=d(b^3-b)$ or $a^3-db^3=a-db$. Let

$$a^3-db^3=e, a-db=e. \quad (3)$$

For formulae (3), we have $(db+e)^3-db^3=e$ or

$$(d^3-d)b^3+3d^2b^2e+3bde^2+e^3-e=0. \quad (4)$$

Suppose that $(b,e)=t$ and $b=tb_1$, $e=te_1$, then formulae (4) becomes

$$(d^3-d)t^3b_1^3+3d^2t^3b_1^2e_1+3t^3db_1e_1^2+t^3e_1^3-te_1=0$$ or

$$(d^3-d)b_1^3+3d^2b_1^2e_1+3db_1e_1^2+e_1^3-\frac{e_1}{t^2}=0. \quad (5)$$

Let $k=\frac{e_1}{t^2}$, then k is an integer, obviously $k|e_1$. Since $(b_1,e_1)=1$, thus $(k,b_1)=1$, from formulae (5) we obtain $k|(d^3-d)$. Note that $b=tb_1$, $e=te_1$ and $k=$

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$\frac{e_1}{t^2}$, hence the first equation of formulae(3) becomes

$$a^3 - db_1^3t^3 = kt^3, \quad (6)$$

From formulae(6), we see that $t | a$. Let $a = ta_1$, thus we obtain

$$a_1^3 - db_1^3 = k. \quad (7)$$

Note that $k | (d^3 - d)$, by lemma 1, the solution of formulae(7) satisfies $|a_1| < C_1, |b_1| < C_1$, where C_1 is an effectively computable constant depending upon d.

Since $a = ta_1, b = tb_1, e = te_1, e_1 = kt^2$, the second equation of formulae(3) gives

$$a_1 - db_1 = kt^2. \quad (8)$$

Therefore $|t| < \frac{(|a_1| + |db_1|)}{|k|}$, hence $|a| = |ta_1| < C, |b| = |tb_1| < C$, where C is an effectively computable constant depending upon d. This proves the theorem 1.

Theorem 2 The only integer solutions of the equation

$$y(y+1)(y+2) = 2x(x+1)(x+2) \quad (9)$$

are given by $(x,y) = (-2,-2), (-2,0), (-2,-1), (0,-1), (0,0), (0,-2), (-1,-1), (-5,-6)$ and $(3,4)$.

Proof Let $d = 2$, then $k | (d^3 - d) = 6$, formulae(7) and formulae(8) give

$$a_1^3 - 2b_1^3 = \pm 1, a_1 - 2b_1 = \pm t^2, \quad (10)$$

$$\text{or } a_1^3 - 2b_1^3 = \pm 2, a_1 - 2b_1 = \pm 2t^2, \quad (11)$$

$$\text{or } a_1^3 - 2b_1^3 = \pm 3, a_1 - 2b_1 = \pm 2t^2, \quad (12)$$

$$\text{or } a_1^3 - 2b_1^3 = \pm 6, a_1 - 2b_1 = \pm 6t^2, \quad (13)$$

The first equation of formulae(10) has only solutions $a_1 = \pm 1, b_1 = 0$ and $a_1 = \mp 1, b_1 = \mp 1$. These give $t^2 = 1, t = \pm 1$, further give $(y,x) = (-2,-1), (0,-1), (-2,-2), (0,0)$ respectively.

The first equation of formulae(11) gives $2 | a_1$, let $a_1 = 2a_2$, hence we have $4a_2^3 - b_1^3 = \pm 1$. By lemma 2, it gives $a_1 = 0, b_1 = \mp 1$, therefore $(y,x) = (-1,-2), (-1,0)$ respectively.

The first equation of formulae(12) has only solutions $(a_1, b_1) = (\pm 1, \mp 1), (\mp 5, \mp 4)$, so $t^2 = 1$. These give solutions $(y,x) = (0,-2), (-2,0), (-6,-5), (4,3)$ respectively.

The first equation of formulae(13) becomes

$$4a_2^3 - b_1^3 = \pm 3, a_1 = 2a_2. \quad (14)$$

From lemma 2, formulae(14) has only solutions $a_2 = \pm 1, b_1 = \pm 1$, and hence $a_1 = \pm 2, b_1 = \pm 1$. Therefore the second equation of formulae(13) gives $\pm 6t^2 = a_1 - 2b_1 = 0$, so $a = ta_1 = 0, b = tb_1 = 0$, this gives $(x,y) = (-1,-1)$ by $a = y+1, b = x+1$. The proof is completed.

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