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Bifurcations of Travelling Wave Solutions for the Generalized Water Wave Equations*

广义水波方程组行波解的分支

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Abstract The bifurcation of travelling wave solutions for the generalized water wave equations are studied by using the bifurcation theory of planar dynamical systems. Under various parameter conditions, all exact explicit formulas of solitary wave solutions and kink(anti-kink) wave solutions and uncountable infinity many periodic wave solutions are listed.

Key words generalized water wave equations, solitary wave, kink wave, anti-kink wave

摘要:应用动力系统分支理论,研究广义水波方程组行波解的分支.在固定的参数条件下给出广义水波方程组的孤立波、扭结(反扭结)波解的精确表达式,并证明该方程组存在不可数无穷多个周期波解.

关键词:广义水波方程组 孤立波 扭结波 反扭结波
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We consider the following generalized water wave equations:

$$u_t + uu_x + v_x = 0,\ v_t + [u(v+1)]_x + \frac{U}{4}u_{xxx} = 0, \quad (0.1)$$

where v is the elevation of the water wave, u is the surface velocity of water along x-direction, U is a non-zero real number. Specially, when U$=1$, equations(0.1) is called nonlinear long wave equations of Boussinesq class[1]. Recently, the new solitary solutions for this equation were constructed by He and Xu[2]. Unfortunately, the results in reference [2] is not complete, since the authors did not study the bifurcation behaviors of phase portraits for the corresponding travelling wave equations. In this paper, we consider bifurcation problem of travelling wave for equation (0.1), by using the bifurcation theory of dynamical system[3,4]. Under fixed parameter conditions, all exact explicit formulas of solitary wave, kink wave and periodic wave solutions can be easily obtained.

We first consider the travelling wave solutions in the form

$$u(x,t) = u(Y), v(x,t) = v(Y), Y = k(x-ct), k \neq 0, \quad (0.2)$$

where c denotes the wave speed. Therefore equation (0.1) reduces to be

$$-cu' + uu' + v' = 0, \quad (0.3)$$

$$-cv' + [u(v+1)]' + \frac{1}{4}Uk^2u''' = 0, \quad (0.4)$$

where "$'$" is the derivative with respect to Y. Integrating formulae(0.3), formulae(0.4) once, we have

$$-cu + \frac{1}{2}u^2 + v = g_1, \quad (0.5)$$

$$-cv + u(v+1) + \frac{1}{4}Uk^2u'' = g_2, \quad (0.6)$$

where g_1, g_2 are integration constant. Inserting (0.5) into formulae(0.6), we have

$$u'' = \frac{4}{Uk^2}[(g_2 + cg_1) + (c^2 - g_1 - 1)u + \frac{1}{2}cu^2 + \frac{1}{2}u^3]. \quad (0.7)$$

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Clearly, formulae(0.5) is equivalent to the following two-dimensional systems:

$$\frac{du}{d\xi}=y,\ \frac{dy}{d\xi}=\frac{2}{Uk^2}[2(g_2+cg_1)+2(c^2-g_1-1)u+cu^2+u^3].\qquad(0.8)$$

1 Bifurcations of phase portraits of systems (0.8)

In this section, we study the phase portraits of system(0.8). We make the transformation $u=\varphi(\xi)+\varphi_*$, $y=y$, $\xi=\xi$, where φ_* satisfies the equation

$$2(g_2+cg_1)+2(c^2-g_1-1)u+cu^2+u^3=0.\qquad(1.1)$$

The system (0.8) becomes

$$\frac{d\varphi}{d\xi}=y,\ \frac{dy}{d\xi}=r\varphi(\varphi^2+p\varphi+q),\qquad(1.2)$$

where $r=\frac{2}{Uk^2}$, $p=c+3\varphi_*$, $q=2(c^2-g_1-1+c\varphi_*+\frac{3}{2}\varphi_*^2)$.

The systems (1.2) has the first integral

$$y^2=\frac{r}{2}\varphi^2(\varphi^2+\frac{4}{3}p\varphi+2q)+h,\qquad(1.3)$$

and

$$H(\varphi,y)=y^2-\frac{r}{2}\varphi^2(\varphi^2+\frac{4}{3}p\varphi+2q)=h.\qquad(1.4)$$

Thus, we have

$$y=\pm(\frac{r}{2}\varphi^2(\varphi^2+\frac{4}{3}p\varphi+2q)+h)^{\frac{1}{2}}.\qquad(1.5)$$

Denote that

$$f_1(\varphi)=\varphi^2+p\varphi+q,\ f_2(\varphi)=\varphi^2+\frac{4}{3}p\varphi+2q,$$

$$\Delta_1=p^2-4q,\ \Delta_2=\frac{16}{9}p^2-8q,$$

which imply the relations in the (p,q)-parameter plane L_1: $q=\frac{1}{4}p^2$, L_2: $q=\frac{2}{9}p^2$.

Thus, we have

(i) If $\Delta_1>0, q\neq 0$, there exist 3 equilibrium points of system (1.2): $A_{1,2}(\frac{-p\pm\sqrt{\Delta_1}}{2},0)$, $O(0,0)$; when $p=0, q<0$, there exist 3 equilibrium points of system (1.2): $A_{1,2}(\pm\sqrt{-q},0)$, $O(0,0)$;

(ii) If $\Delta_1=0, p\neq 0$, there exist 2 equilibrium points of system (1.2): $O(0,0)$, $A(-\frac{p}{2},0)$; when $p=q=0$, there exist a tri-root equilibrium point of system (1.2): $O(0,0)$;

(iii) If $\Delta_1<0$, there exists an equilibrium point of system(1.2): $O(0,0)$.

For $H(\varphi,y)$ defined by formulae(1.4), we have $h_i=H(\varphi_i,0)=-\frac{r}{2}\varphi_i^2(\frac{1}{3}p\varphi_i+q)$, $i=1,2,3$.

For a fixed h, the level curve $H(\varphi,y)=h$ defined by formulae(1.4) determines a set of invariant curves of system(1.2), which contains different branches of curves. As h is varied, it defines different families of orbits of system (1.2) with different dynamical behaviors.

Following, we consider the bifurcations of the phase portraits of system (1.2). In the (p,q)-parameter plane, the curves L_1, L_2, L_3: $p=0(q<0)$ and the straight line L_4: $q=0$ partition it into 7 regions shown in Fig. 1.

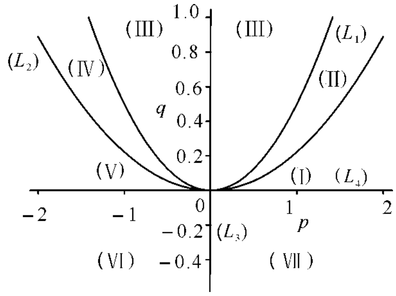

Fig. 1 The bifurcation set of system(1.2) in (p,q)-parameter plane.

From the above analysis we obtain the different phase portraits of system(1.2) shown in Fig. 2 and Fig. 3.

2 Solitary wave and kink (or anti-kink) wave solutions determined by equation(0.1)

In this section, we shall give all exact explicit parametric representations of solitary wave solutions and kink (or anti-kink) wave solutions of equation (0.1) under given parameter conditions.

(i) The case $r>0$.

(1) $(p,q)\in$ (Ⅰ) or $(p,q)\in$ (Ⅴ). In this case, we have the phase portrait of system(1.2) shown in Figs. 2(a) and (e). Notice that $H(0,0)=0=h_2$. We see from formulae(1.4) that the arch curve connecting $O(0,0)$ in the right side of the straight line $\varphi=0$ has the algebraic equation

$$y=\pm(\frac{r}{2}\varphi^2(\varphi^2+\frac{4}{3}p\varphi+2q))^{\frac{1}{2}}.\qquad(2.1)$$

Thus, by using the first equation of system(1.2) and formulae(2.1), we obtain the following solitary wave solution with valley form of equation(0.1).

$$u(x,t)=\varphi+\varphi_*,\ v(x,t)=g_1+c(\varphi+\varphi_*)-\frac{1}{2}(\varphi+\varphi_*)^2,\qquad(2.2)$$

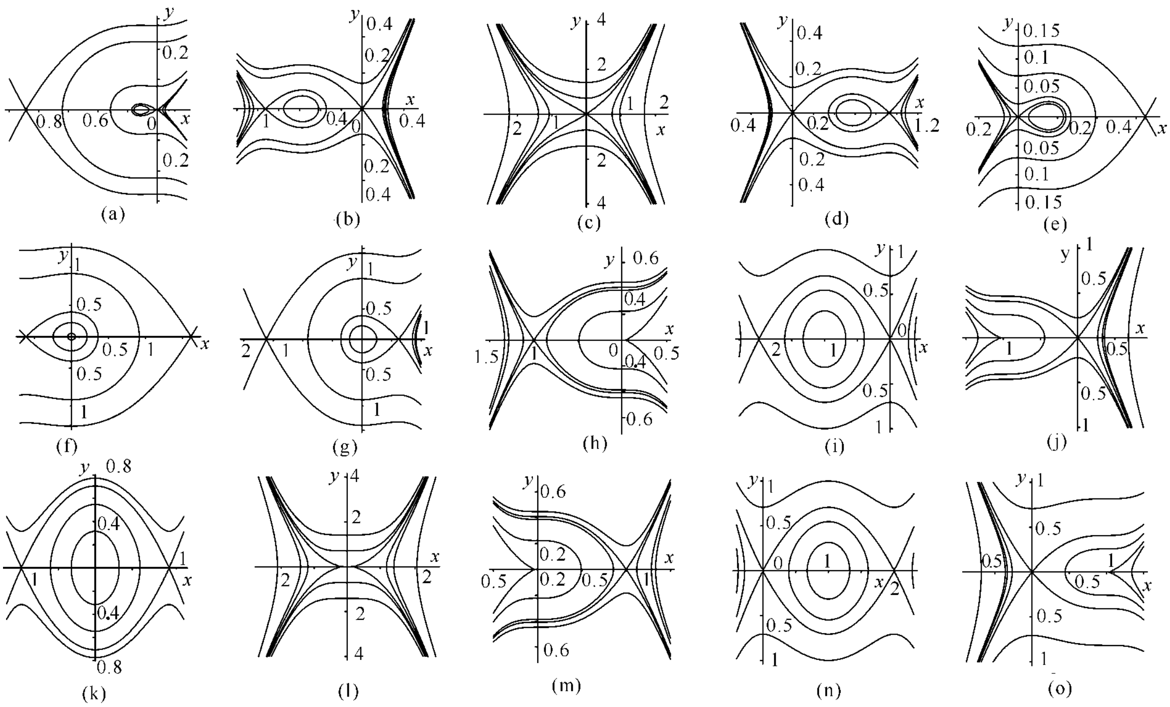

Fig. 2 The phase portraits of system (1.2) for $r > 0$

(a) $(p,q) \in$ (Ⅰ), (b) $(p,q) \in$ (Ⅱ), (c) $(p,q) \in$ (Ⅲ), (d) $(p,q) \in$ (Ⅳ), (e) $(p,q) \in$ (Ⅴ), (f) $(p,q) \in$ (Ⅵ), (g) $(p,q) \in$ (Ⅶ), (h) $(p,q) \in (L_4), p > 0$, (i) $(p,q) \in (L_2), p > 0$, (j) $(p,q) \in (L_1), p > 0$, (k) $(p,q) \in (L_3)$, (l) $(p,q) = (0,0)$, (m) $(p,q) \in (L_4), p < 0$, (n) $(p,q) \in (L_2), p < 0$, (o) $(p,q) \in (L_1), p > 0$.

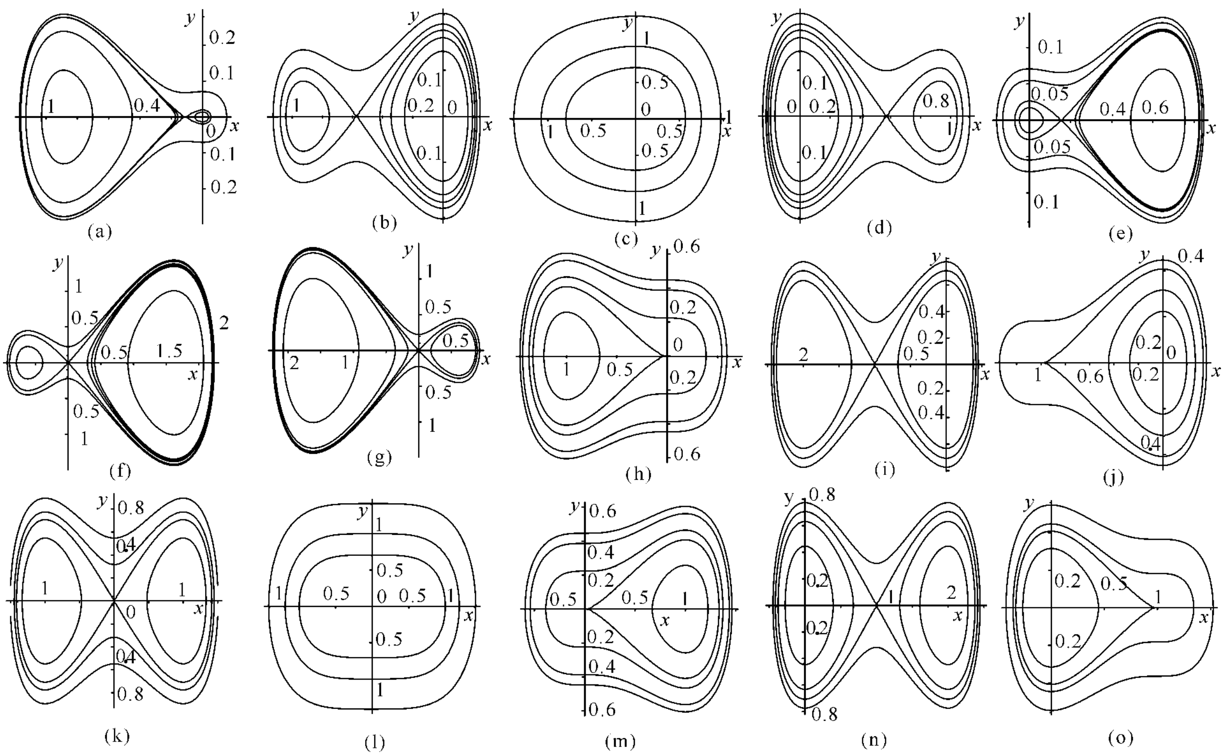

Fig. 3 The phase portraits of system (1.2) for $r < 0$

(a) $(p,q) \in$ (Ⅰ), (b) $(p,q) \in$ (Ⅱ), (c) $(p,q) \in$ (Ⅲ), (d) $(p,q) \in$ (Ⅳ), (e) $(p,q) \in$ (Ⅴ), (f) $(p,q) \in$ (Ⅵ), (g) $(p,q) \in$ (Ⅶ), (h) $(p,q) \in (L_4), p > 0$, (i) $(p,q) \in (L_2), p > 0$, (j) $(p,q) \in (L_1), p > 0$, (k) $(p,q) \in (L_3)$, (l) $(p,q) = (0,0)$, (m) $(p,q) \in (L_4), p < 0$, (n) $(p,q) \in (L_2), p < 0$, (o) $(p,q) \in (L_1), p < 0$.

where $\varphi = 3q/(-p \pm \sqrt{p^2 - \frac{9}{2}q\cosh(\sqrt{qr}(x - ct))})$.

(2) $(p,q)\in$ (II) or $(p,q)\in$ (VI). In this case, we have the phase portrait of system (1.2) shown in Figs. 2(b) and (f). Notice that $H(\varphi_1 = -\frac{1}{2}(p + \sqrt{\Delta_1}), 0) = -\frac{r}{2}\varphi_1^3(\frac{1}{3}p\varphi_1 + q) = h_1$. We see from formulae(1.4) that the arch curve connecting $A_1(\varphi_1, 0)$ has the algebraic equation

$$y = \pm\sqrt{\frac{r}{2}}(\varphi - \varphi_1)[\varphi^3 - \frac{2}{\varphi_1}(\frac{1}{3}p\varphi_1 + q)\varphi - (\frac{1}{3}p\varphi_1 + q)]^{\frac{1}{2}}. \qquad (2.3)$$

Thus, by using the first equation of system (1.2) and formulae(2.3), we obtain the following solitary wave solution with peak form of equation(0.1).

$$u(x,t) = \varphi + \varphi_*, v(x,t) = g_1 + c(\varphi + \varphi_*) - \frac{1}{2}(\varphi + \varphi_*)^2, \qquad (2.4)$$

where $\varphi = \varphi_1 + (p\varphi_1 + 2q)/(\varphi_1 + \frac{1}{3}p \pm \frac{1}{6}\sqrt{p(7p + 3\sqrt{\Delta_1})}\cosh(\sqrt{-r(p\varphi_1 + 2q)}(x - ct)))$.

(3) $(p,q)\in$ (III) or $(p,q)\in$ (VII). In this case, we have the phase portrait of system (1.2) shown in Figs. 2(c) and (g). Notice that $H(\varphi_3 = \frac{1}{2}(-p + \sqrt{\Delta_1}), 0) = -\frac{r}{2}\varphi_3^3(\frac{1}{3}p\varphi_3 + q) = h_3$. Similar to the cases (2), we obtain the following solitary wave solutions with valley form of equation(0.1).

$$u(x,t) = \varphi + \varphi_*, v(x,t) = g_1 + c(\varphi + \varphi_*) - \frac{1}{2}(\varphi + \varphi_*)^2, \qquad (2.5)$$

where

$$\varphi = \varphi_3 + (p\varphi_3 + 2q)/(\varphi_3 + \frac{1}{3}p \pm \frac{1}{6}\sqrt{p(7p - 3\sqrt{\Delta_1})}\cosh(\sqrt{-r(p\varphi_3 + 2q)}(x - ct))).$$

(4) $(p,q)\in(L_2)$, $pq\neq 0$. In this case, we have the phase portrait of system (1.2) shown in Figs. 2(i) and (n). Notice that $H(0,0) = 0 = h_1 = h_3$. Similar to the cases (1), we obtain the following kink and anti-kink wave solutions of equation(0.1).

$$u(x,t) = \varphi + \varphi_*, v(x,t) = g_1 + c(\varphi + \varphi_*) - \frac{1}{2}(\varphi + \varphi_*)^2, \qquad (2.6)$$

where

$$\varphi = 4p/3(\cosh(\frac{2p}{3}\sqrt{\frac{r}{2}}(x - ct)) \pm \sinh(\frac{2p}{3}\sqrt{\frac{r}{2}}(x - ct)) - 4p).$$

(5) $(p,q)\in(L_3)$. In this case, we have the phase portrait of system (1.2) shown in Fig. 2(k). Notice that $H(\pm\sqrt{-q}, 0) = \frac{1}{2}rq^2 = h_1$, we obtain the following kink and anti-kink wave solutions of equation(0.1).

$$u(x,t) = \varphi + \varphi_*, v(x,t) = g_1 + c(\varphi + \varphi_*) - \frac{1}{2}(\varphi + \varphi_*)^2, \qquad (2.7)$$

where $\varphi = \pm\sqrt{-q}\cosh(\frac{1}{2}\sqrt{2(-q)r}(x - ct))$.

(ii) The case $r < 0$.

(1) $(p,q)\in$ (I) or $(p,q)\in$ (II). In this case, we have the phase portrait of system (1.2) shown in Figs. 3(a) and (b). Notice that $H(\varphi_2 = \frac{1}{2}(-p + \sqrt{\Delta_1}), 0) = -\frac{r}{2}\varphi_2^3(\frac{1}{3}p\varphi_2 + q) = h_2$. We see from formulae(1.4) that the arch curve connecting $A_2(\varphi_2, 0)$ has the algebraic equation

$$y = \pm\sqrt{\frac{-r}{2}}(\varphi - \varphi_2)[-\varphi^3 + \frac{2}{\varphi_2}(\frac{1}{3}p\varphi_2 + q)\varphi + (\frac{1}{3}p\varphi_2 + q)]^{\frac{1}{2}}. \qquad (2.8)$$

Thus, by using the first equation of system (1.2) and formulae(2.8), we obtain the following solitary wave solutions with valley and peak form of equation(0.1).

$$u(x,t) = \varphi + \varphi_*, v(x,t) = g_1 + c(\varphi + \varphi_*) - \frac{1}{2}(\varphi + \varphi_*)^2, \qquad (2.9)$$

where

$$\varphi = \varphi_2 + (p\varphi_2 + 2q)/(\varphi_2 + \frac{1}{3}p \pm \frac{1}{6}\sqrt{p(7p - 3\sqrt{\Delta_1})}\cosh(\sqrt{-r(p\varphi_2 + 2q)}(x - ct))).$$

(2) $(p,q)\in$ (IV) or $(p,q)\in$ (V). In this case, we have the phase portrait of system (1.2) shown in Figs. 3(d) and (e). Notice that $H(\varphi_2 = -\frac{1}{2}(p + \sqrt{\Delta_1}), 0) = -\frac{r}{2}\varphi_2^3(\frac{1}{3}p\varphi_2 + q) = h_2$. We obtain the solitary wave solutions with valley and peak form of

equation(0. 1), which is the same as formulae(2. 9).

(3) $(p,q)\in$ (Ⅵ) or $(p,q)\in$ (Ⅶ). In this case, we have the phase portrait of system(1. 2) shown in Figs. 3(f) and (g). Notice that $H(0,0)=0=h_2$. We see from formulae(1. 4) that the arch curve connecting $O(0,0)$ has the algebraic equation

$$y=\pm\left(\frac{-r}{2}\varphi(-\varphi^3-\frac{4p}{3}\varphi-2q)\right)^{\frac{1}{2}}. \qquad (2.10)$$

Thus, by using the first equation of system(1. 2) and (2. 10), we obtain the following solitary wave solutions with valley and peak form of equation(0. 1).

$$u(x,t)=\varphi+\varphi_*, v(x,t)=g_1+c(\varphi+\varphi_*)-\frac{1}{2}(\varphi+\varphi_*)^2, \qquad (2.11)$$

where $\varphi=3q/(-p\pm\sqrt{p^2-\frac{9}{2}q}\cosh(\sqrt{qr}(x-ct)))$.

(4) $(p,q)\in(L_4)$. In this case, we have the phase portrait of system(1. 2) shown in Figs. 3(h) and (m). Notice that $H(0,0)=0$, we obtain the following solitary wave solutions with valley and peak form of equation(0. 1).

$$u(x,t)=\varphi+\varphi_*, v(x,t)=g_1+c(\varphi+\varphi_*)-\frac{1}{2}(\varphi+\varphi_*)^2, \qquad (2.12)$$

where $\varphi=\frac{12p}{9-2rp^2(x-ct)^2}$.

(5) $(p,q)\in(L_2), p\neq 0$. In this case, we have the phase portrait of system(1. 2) shown in Figs. 3(i) and (n). Notice that $H(\varphi=-\frac{1}{3}p,0)=-\frac{r}{162}p^4=h_2$, we obtain the following solitary wave solutions with valley and peak form of equation(0. 1).

$$u(x,t)=\varphi+\varphi_*, v(x,t)=g_1+c(\varphi+\varphi_*)-\frac{1}{2}(\varphi+\varphi_*)^2, \qquad (2.13)$$

where $\varphi=-\frac{1}{3}p\pm\sqrt{2}p/3\cosh(\sqrt{-r}(x-ct))$.

(6) $(p,q)\in(L_1), p\neq 0$. In this case, we have the phase portrait of system(1. 2) shown in Figs. 3(j) and (o). Notice that $H(-\frac{1}{2}p,0)=-\frac{r}{96}p^4$, we obtain the following solitary wave solutions with valley and peak form of equation(0. 1).

$$u(x,t)=\varphi+\varphi_*, v(x,t)=g_1+c(\varphi+\varphi_*)-\frac{1}{2}(\varphi+\varphi_*)^2, \qquad (2.14)$$

where $\varphi=12p/(9-2rp^2(x-ct)^2)$.

(7) $(p,q)\in(L_3)$. In this case, we have the phase portrait of system(1. 2) shown in Figs. 3(k). Notice that $H(0,0)=0$, we obtain the following solitary wave solutions with valley and peak form of equation(0. 1).

$$u(x,t)=\varphi+\varphi_*, v(x,t)=g_1+c(\varphi+\varphi_*)-\frac{1}{2}(\varphi+\varphi_*)^2, \qquad (2.15)$$

where $\varphi=\pm\sqrt{2(-q)}\sinh(\sqrt{qr}(x-ct))$.

3 Periodic wave solutions determined by equation(0. 1)

In this section, we use the results of section 1 to discuss the existence of uncountable infinity many periodic wave solutions.

Theorem 4. 1 The case $r>0$.

(1) Corresponding to Figs. 2(a), (b), (d), (e), (f) and (g), equation(0. 1) has one family of smooth periodic wave solutions.

(2) Corresponding to Fig. 2(k). Suppose that $h\in(0,h_3)$, equation(0. 1) has one family of smooth periodic wave solutions.

(3) Corresponding to Figs. 2(i) and formulae (2. 14). Suppose that $h\in(h_2,0)$, equation(0. 1) has one family of smooth periodic wave solutions.

Theorem 4. 2 The case $r<0$.

(1) Corresponding to Fig. 3(l), equation(0. 1) has one family of smooth periodic wave solutions.

(2) Corresponding to Fig. 3(k).

(i) Suppose that $h\in(h_1,0)$, equation(0. 1) have two families of smooth periodic wave solutions.

(ii) Suppose that $h\in(0,+\infty)$, equation(0. 1) has one family of smooth periodic wave solutions.

(3) Corresponding to Fig. 3(c), equation(0. 1) has one family of smooth periodic wave solutions.

(4) Corresponding to Fig. 3(f).

(i) Suppose that $h\in(h_3,h_1)$, equation(0. 1) has one family of smooth periodic wave solutions.

(ii) Suppose that $h\in(0,+\infty)$, equation(0. 1) has one family of global periodic wave solutions.

(iii) Suppose that $H(\varphi,y)=h_1$, equation(0. 1) has one smooth periodic wave solution.

(iv) Suppose that $h\in(h_1,0)$, equation(0. 1) has two families of smooth periodic wave solutions.

(5) Corresponding to Fig. 3(g).

(i) Suppose that $h\in(h_1,h_3)$, equation(0. 1) has one family of smooth periodic wave solutions.

(ii) Suppose that $h \in (0, +\infty)$, equation (0.1) has one family of global smooth periodic wave solutions.

(iii) Suppose that $H(\varphi, y) = h_3$, equation (0.1) has one periodic travelling wave solution.

(iv) Suppose that $h \in (h_3, 0)$, equation (0.1) has two families of smooth periodic wave solutions.

(6) Corresponding to Figs. 3(i) and (n).

(i) Suppose that $h \in (0, h_2)$, equation (0.1) have two families of smooth periodic wave solutions.

(ii) Suppose that $h \in (h_2, +\infty)$, equation (0.1) has one family of smooth periodic wave solutions.

(7) Corresponding to Figs. 3(j) and (o). Suppose that $h \in (0, h_2)$ or $h \in (h_2, +\infty)$, equation (0.1) has two families of smooth periodic wave solutions.

(8) Corresponding to Figs. 3(h) and (m). Suppose that $h \in (h_1, 0)$ or $h \in (h_3, 0)$ and $h \in (0, +\infty)$, equation (0.1) have two families of smooth periodic wave solutions.

(9) Corresponding to Figs. 3(a) and (b), by using the transformation $\tilde{\varphi} = \varphi - \frac{-p + \sqrt{\Delta_1}}{2}$, similar to the cases (5), we obtain all parametric representations of the periodic wave solutions of equation (0.1).

(10) Corresponding to Figs. 3(d) and (e), similar to the cases (4), we obtain all parametric representations of the periodic wave solutions of equation (0.1).

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西双版纳热带植物园入侵生态学研究取得新进展

生物入侵是严重的社会经济和环境问题,是全球变化的重要组成部分,入侵种的扩散严重威胁着全球生物多样性安全、生态系统的结构和功能、农林牧业生产。尽管入侵种的危害越来越严重,对其研究越来越多,但是目前有关生物入侵的理论和假说还不能说明为什么入侵种具有如此强的入侵能力。增强竞争能力的进化(EICA)假说是最有影响的入侵理论之一,该假说认为入侵地的天敌缺乏将使外来种通过进化降低天敌防御能力,而把原来用于防御的资源用于生长和繁殖等过程,从而提高竞争力能,促进入侵。该假说的前提是植物的防御是有成本的,生长和防御有权衡关系,而这种权衡关系也已被很多研究所证实。但是,最近科学家研究发现,在西双版纳热带植物园外来入侵植物斑点矢车菊 (*Centaurea maculosa*) 入侵种群的生长速度和竞争能力高于原产地种群,但是入侵种群高的竞争能力并不是以天敌防御能力降低为代价的。相反,入侵种群对广普天敌和专性天敌的防御能力要高于原产地种群,它们不仅能更好地抑制天敌取食(抗性),也能更好地忍耐天敌的攻击(耐性)。入侵种群高的天敌防御能力与其高的化学防御物质前体含量、坚韧的叶片和多的叶毛有关。入侵种群的这些特性是由遗传因素决定的。这项研究表明,增强竞争能力的进化并不总是由能量或资源向生长和防御分配的权衡关系导致的。这与传统的生态学理论不同,与EICA假说也不同,与冯玉龙等提出的氮分配进化假说也不一致,必将促进今后相关领域的研究。

(据科学时报)