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The Optimal Approximation Solution of Matrix Inverse Problems for D-symmetric Semidefinite Matrices*

D对称半正定矩阵反问题的最佳逼近解

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Abstract For given $A^* \in R^{n\times n}$, when the set of solution of the matrix equation $AX = B$ in $D^{-2}SR_0^{n\times n}$ is nonempty, the optimal approximation solution of A^* in S_A is given, and a numerical example is performed to illustrate the validity of the optimal approximation solution.

Key words matrix, D-symmetric matrix, semidefinite, inverse problems, optimal approximation

摘要：对任意矩阵 $A^* \in R^{n\times n}$，当矩阵方程 $AX = B$ 在 D对称半正定矩阵集 $D^{-2}SR_0^{n\times n}$ 中的解集 S_A 非空时，给出 A^* 在 S_A 中的最佳逼近解，并用数值算例验证最佳逼近解的有效性。

关键词：矩阵 D对称矩阵 半正定 反问题 最佳逼近
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Matrix inverse problems and its optimal approximation problems have been widely used in control theory, vibration theory, civil structure engineering, nonlinear programming and the other fields. There are many results[1~4].

The following notations are used throughout this paper.

Let $R^{n\times m}$ denotes the set of $n\times m$ real matrices, $OR^{n\times n}$ denotes the set of $n\times n$ real orthogonal matrices, $SR^{n\times n}$ denotes the set of $n\times n$ real symmetric matrices. $R_0^{n\times m}$ denotes the set of $n\times m$ real semidefinite (need not symmetric) matrices. A^+ denotes Moore-Penrose generalized inverse of matrix A. We denote the set of $n\times n$ real positive definite symmetric matrices and real semidefinite symmetric matrices by $SR_+^{n\times n}$ and $SR_0^{n\times n}$ respectively, that is

$$SR_+^{n\times n} = \{A \mid A \in SR^{n\times n}, X^{\mathrm{T}}AX > 0, \forall X \in R^n, X \neq 0\},$$

$$SR_0^{n\times n} = \{A \mid A \in SR^{n\times n}, X^{\mathrm{T}}AX \geqslant 0, \forall X \in R^n\},$$

and denote the set of $n\times n$ real negative semidefinite symmetric matrices by $SR_-^{n\times n}$, that is,

$$SR_-^{n\times n} = \{A \mid A \in SR^{n\times n}, X^{\mathrm{T}}AX \leqslant 0, \forall X \in R^n\}.$$

One denotes by $A > 0$ (or $A \geqslant 0$) $A \in SR_+^{n\times n}$ (or $A \in SR_0^{n\times n}$).

Definition 0.1[5] Given $D = \mathrm{diag}(d_1, d_2, \cdots, d_n) \in R^{n\times n}, d_i > 0, i = 1, 2, \cdots, n$, for $A \in R^{n\times n}$, A is called a D-symmetric matrix if $D^2A = SR^{n\times n}$.

We denote the set of all D-symmetric matrices by $D^{-2}SR^{n\times n}$.

Definition 0.2[6] The set $D^{-2}SR_0^{n\times n}$ is defined as follows:

$$D^{-2}SR_0^{n\times n} = \{A \in D^{-2}SR^{n\times n} \mid X^{\mathrm{T}}D^2AX \geqslant 0, \forall X \in R^n\}.$$

Let $D^{-2}SR_-^{n\times n} = \{A \in D^{-2}SR^{n\times n} \mid X^{\mathrm{T}}D^2AX \leqslant 0, \forall X \in R^n\}$.

A is called a D-semidefinite symmetric matrix if $A \in D^{-2}SR_0^{n\times n}$, and A is called a D-negative semidefinite symmetric matrix if $A \in D^{-2}SR_-^{n\times n}$.

It is easy to verify that $D^{-2}SR_0^{n\times n} \subseteq R_0^{n\times n}$, and when $D = I_n$, $D^{-2}SR_0^{n\times n} = R_0^{n\times n}$. Therefore, $SR_0^{n\times n} \subseteq D^{-2}SR_0^{n\times n}$.

In reference [7], we studied the following matrix

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inverse problem concerned with D-semidefinite symmetric matrix.

Problem Ⅰ Given $X, B \in R^{n \times m}$, find the condition for the solvability of the matrix equation

$$AX = B, \qquad (0.1)$$

and the general forms of solution A in the set of D-semidefinite symmetric matrix.

Because D-semidefinite symmetric matrix is a kind of matrices within semidefinite symmetric matrix and semidefinite (need not symmetric) matrix, it is very meaningful to study this kind of matrices.

Based on reference [7], the optimal approximation problems and its numerical solution for D-semidefinite symmetric matrices are discussed in this paper. The main problem is described as follows

Problem Ⅱ Given $A^* \in R^{n \times n}$, find $A \in S_A$, such that

$$\| A - A^* \| = \min_{A \in S_A} \| A - A^* \|, \qquad (0.2)$$

where S_A is the set of solutions of Problem I, $\| \cdot \|$ is Frobenius norm.

1 Lemmas

Given matrices $X, B \in R^{n \times m}$, $D = \mathrm{diag}(d_1, d_2, \cdots, d_n) \in R^{n \times n}$, $d_i > 0, i = 1, 2, \cdots, n$. Let the singular value decomposition (SVD) of the matrix D^2X be

$$D^2X = U\begin{pmatrix} \Sigma & O \\ O & O \end{pmatrix} V^T = U_1 \Sigma V_1^T, \qquad (1.1)$$

where $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_k)$, $\sigma_i > 0, i = 1, 2, \cdots, k$, $k = \mathrm{rank}(X)$, $U = (U_1, U_2) \in OR^{n \times n}$, $V = (V_1, V_2) \in OR^{m \times m}$, $U_1 \in R^{n \times k}$, $V_1 \in R^{m \times k}$. Suppose the partition of $\bar{B} = U^T B V$ is

$$\bar{B} = U^T B V = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \qquad (1.2)$$

where $B_{11} \in R^{k \times k}$, $B_{12} \in R^{k \times (m-k)}$, $B_{21} \in R^{(n-k) \times k}$, $B_{22} \in R^{(n-k) \times (m-k)}$.

Lemma 1.1[7] Suppose that the SVD of the matrix D^2X is the form of formulae(1.1) and the partition of $\bar{B} = U^T B V$ is given by formulae(1.2). Then Problem I is solvable in $D^{-2}SR_0^{n \times n}$ if and only if

$$B_{12} = O, B_{22} = O, \qquad (1.3)$$

$$B_{11}\Sigma^{-1} \in SR_0^{k \times k}, \qquad (1.4)$$

$$\mathrm{rank}(B_{11}\Sigma^{-1}\Sigma^{-1}B_{21}^T) = \mathrm{rank}(B_{11}). \qquad (1.5)$$

Moreover, the general expression of solution of Problem Ⅰ is

$$A = U\begin{pmatrix} B_{11}\Sigma^{-1} & \Sigma^{-1}B_{21}^T \\ B_{21}\Sigma^{-1} & B_{21}\Sigma^{-1}(B_{11}\Sigma^{-1})^+ \Sigma^{-1}B_{21}^T \end{pmatrix} U^T \cdot D^2 + U\begin{pmatrix} O & O \\ O & G \end{pmatrix} U^T D^2, \qquad (1.6)$$

where $G \in SR_0^{(n-k) \times (n-k)}$.

Lemma 1.2[8] Given a nonempty closed convex cone $S \subseteq R^{n \times n}$, and $F \in R^{n \times n}$, $\Delta = \mathrm{diag}(\delta_1, \delta_2, \cdots, \delta_n)$, $\delta_i > 0, i = 1, 2, \cdots, n$. Then there exists a unique optimal approximation $E \in S$ such that

$$\| (E - F)\Delta \| = \min_{E \in S} \| (E - F)\Delta \|. \qquad (1.7)$$

2 The solution of problem Ⅱ

Theorem 2.1 Suppose conditions of Lemma 1.1 hold. Let

$$A_0 = U\begin{pmatrix} B_{11}\Sigma^{-1} & \Sigma^{-1}B_{21}^T \\ B_{21}\Sigma^{-1} & B_{21}\Sigma^{-1}(B_{11}\Sigma^{-1})^+ \Sigma^{-1}B_{21}^T \end{pmatrix} U^T D^2, \qquad (2.1)$$

and

$$E = U_2 G U_2^T, F = (A^* - A_0)D^{-2}, \Delta = D^2. \qquad (2.2)$$

Then there exists a unique optimal approximation A in S_A such that formulae(0.2) holds, and

$$A = A_0 + U\begin{pmatrix} O & O \\ O & G \end{pmatrix} U^T D^2, \qquad (2.3)$$

where $G = U_2^T E U_2$, and E is the unique optimal approximation solution of the least-squares problem

$$\| (E - F)\Delta \| = \min, E \in SR_0^{n \times n},$$

that is,

$$\| (E - F)\Delta \| = \min_{E \in SR_0^{n \times n}} \| (E - F)\Delta \|. \qquad (2.4)$$

Proof For any $A \in S_A$, one has

$$S_A = \left\{ A_0 + U\begin{pmatrix} O & O \\ O & G \end{pmatrix} U^T D^2 \mid G \in SR_0^{(n-k) \times (n-k)} \right\}. \qquad (2.5)$$

Thus,

$$\| A - A^* \|^2 = \| A_0 + U\begin{pmatrix} O & O \\ O & G \end{pmatrix} U^T D^2 - A^* \|^2 = \| U\begin{pmatrix} O & O \\ O & G \end{pmatrix} U^T D^2 - (A^* - A_0) \|^2 = \| [U\begin{pmatrix} O & O \\ O & G \end{pmatrix} U^T - (A^* - A_0)D^{-2}] D^2 \|^2.$$

Since $U = (U_1, U_2)$ be an orthogonal matrix and the properties of Frobenius norm, one has

$$\| A - A^* \|^2 = \| [U_2 G U_2^T - (A^* -$$

$A_0)D^{-2}]D^2\|^2$.

By formulae(2.2), one has

$$\|A-\hat{A}^*\|=\|(E-F)\Delta\|. \qquad (2.6)$$

Since $SR_0^{n\times n}$ is a nonempty closed convex cone of $R^{n\times n}$, by Lemma 1.2, it is easy to see that there exists a unique matrix $E\in SR_0^{n\times n}$ such that

$$\|(E-F)\Delta\|=\min_{E\in SR_0^{n\times n}}\|(E-F)\Delta\|. \qquad (2.7)$$

Let $G=U_2^{\mathrm{T}}EU_2$ and $A=A_0+U\begin{pmatrix}O & O\\ O & G\end{pmatrix}U^{\mathrm{T}}D^2$. Then A is the solution of Problem II.

3 Algorithm and numerical example

Now based on the method for solving least-squares problem of matrix equation $AX=B$ in $SR^{n\times n}$, we can describe an algorithm for solving Problem II as follows.

Given matrices $A,B\in R^{n\times m}$, $D=\mathrm{diag}(d_1,d_2,\cdots,d_n)$, $d_i>0$, $i=1,2,\cdots,n$. Suppose there are solutions of Problem I in $D^{-2}SR_0^{n\times n}$ and solution set S_A is defined in formulae(2.5).

Remark The key for solving numerical solution of Problem II is how to solve $E\in SR_0^{n\times n}$ such that formulae (2.7) holds. The reference [9] has introduced a kind of algorithm that is convergent and given MATLAB program about it. The algorithm for solving Problem II is described as follows.

Algorithm 1

Step 1: Compute $F=(A^*-A_0)D^{-2}$, $\Delta=D^2$.

Step 2: Based on method in reference [9], find $E\in SR_0^{n\times n}$ such that

$$\|E\Delta-F\Delta\|=\min_{E\in SR_0^{n\times n}}\|E\Delta-F\Delta\|.$$

Step 3: Compute $G=U_2^{\mathrm{T}}EU_2$, then G in $SR_0^{(n-k)\times(n-k)}$ and the solution of Problem II is

$$A=A_0+U\begin{pmatrix}O & O\\ O & G\end{pmatrix}U^{\mathrm{T}}D^2.$$

Example 1 Given matrices $X,B\in R^{5\times 4}$, $D\in R^{5\times 5}$ as follows:

$$X=\begin{pmatrix}1&0&1&0\\0&1&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$B=\begin{pmatrix}\frac{1+\sqrt3}{2\sqrt3}&\frac{1-\sqrt3}{2\sqrt3}&\frac{1}{\sqrt3}&0\\ \frac{1-\sqrt3}{2\sqrt3}&\frac{1+\sqrt3}{2\sqrt3}&\frac{1}{\sqrt3}&0\\ 0&0&0&0\\ \frac{1}{\sqrt3}&\frac{1}{\sqrt3}&\frac{2}{\sqrt3}&0\\ \frac{1}{\sqrt2}&-\frac{1}{\sqrt2}&0&0\end{pmatrix},$$

$D=\mathrm{diag}(0.1,0.1,0.2,0.5,0.6)$.

It is easy to verify that the matrix inverse problem $AX=B$ has solutions in $D^{-2}SR_0^{n\times n}$, and the solution set is

$$S_A=\{A_0+U\begin{pmatrix}O&O\\O&G\end{pmatrix}U^{\mathrm{T}}D^2 \mid G\in SR_0^{3\times 3}\},$$

where

$$A_0=\begin{pmatrix}\frac{1+\sqrt3}{2\sqrt3}&\frac{1-\sqrt3}{2\sqrt3}&0&\frac{25}{\sqrt3}&\frac{36}{\sqrt2}\\ \frac{1-\sqrt3}{2\sqrt3}&\frac{1+\sqrt3}{2\sqrt3}&0&\frac{25}{\sqrt3}&-\frac{36}{\sqrt2}\\ 0&0&0&0&0\\ \frac{1}{\sqrt3}&\frac{1}{\sqrt3}&0&\frac{50}{\sqrt3}&0\\ \frac{1}{\sqrt2}&-\frac{1}{\sqrt2}&0&0&36\end{pmatrix},$$

and

$$U=\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&0&0&0\\ \frac{1}{\sqrt2}&-\frac{1}{\sqrt2}&0&0&0\\ 0&0&\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}&0\\ 0&0&\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&0\\ 0&0&0&0&1\end{pmatrix},$$

$$V=\begin{pmatrix}\frac{1}{\sqrt6}&\frac{1}{\sqrt2}&\frac{1}{\sqrt3}&0\\ \frac{1}{\sqrt6}&-\frac{1}{\sqrt2}&\frac{1}{\sqrt3}&0\\ \frac{2}{\sqrt6}&0&-\frac{1}{\sqrt3}&0\\ 0&0&0&1\end{pmatrix}.$$

Let

$$A^*=\begin{pmatrix}0.113&-0.853&1.213&0.755&-0.264\\ -0.732&1.256&-0.385&0.814&0.321\\ 0.645&-0.213&0.356&1.214&0.876\\ 1.214&0.832&-0.726&0.421&0.568\\ -0.632&1.120&-0.326&0.527&1.231\end{pmatrix}\in R^{5\times 5}.$$

Find $\hat{A}\in S_A$ such that

$$\| A - A^{*} \| = \min_{A \in S_A} \| A - A^{*} \|.$$

Using Algorithm 1, one can obtains $E \in SR_0^{5 \times 5}$, and

$$E = \begin{pmatrix} 424.4835 & 15.8588 & 10.0580 & -35.4363 & -14.6015 \\ 15.8588 & 694.3336 & -5.3892 & -40.3866 & 45.2367 \\ 10.0580 & -5.3892 & 12.4779 & -0.2686 & -0.5451 \\ -35.4363 & -40.3866 & -0.2686 & 5.1627 & -1.3553 \\ -14.6015 & 45.2367 & -0.5451 & -1.3553 & 3.5263 \end{pmatrix},$$

such that

$$\| E\Delta - F\Delta \| = \min_{E \in SR_0^{n \times n}} \| E\Delta - F\Delta \|.$$

By direct calculation, one has

$$G = U_2^{T} E U_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{T} \cdot \begin{pmatrix} 424.4835 & 15.8588 & 10.0580 & -35.4363 & -14.6015 \\ 15.8588 & 694.3336 & -5.3892 & -40.3866 & 45.2367 \\ 10.0580 & -5.3892 & 12.4779 & -0.2686 & -0.5451 \\ -35.4363 & -40.3866 & -0.2686 & 5.1627 & -1.3553 \\ -14.6015 & 45.2367 & -0.5451 & -1.3553 & 3.5263 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 8.5517 & -3.6576 & -1.3438 \\ -3.6576 & 9.0889 & -0.5729 \\ -1.3438 & -0.5729 & 3.5263 \end{pmatrix}.$$

Thus, the solution of Problem II is

$$A = A_0 + U \begin{pmatrix} O & O \\ O & G \end{pmatrix} U^{T} D^{2} = \begin{pmatrix} 0.7887 & -0.2113 & 0.0000 & 14.4338 & 25.4558 \\ -0.2113 & 0.7887 & 0.0000 & 14.4338 & -25.4558 \\ 0.0000 & 0.0000 & 0.4991 & -0.0672 & -0.1962 \\ 0.5774 & 0.5774 & -0.0107 & 30.1582 & -0.4879 \\ 0.7071 & -0.7071 & -0.0218 & -0.3388 & 37.2695 \end{pmatrix}$$

and $\min_{A \in S_A} \| A - A^{*} \| = 48.6474.$

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近地小行星的组成分布

科学家对近地小行星所作的新的光谱测量显示,正如人们可能预料的,近地小行星的总体组成与降落到地球上的最常见的陨石——普通球粒状陨石相似。但是它们的组成分布却是出乎意料的。大约 2/3的近地小行星,包括那些最有可能撞击地球的小行星,与被称为 LL球粒状陨石的一组陨石相符,后者仅占降落到地球上的全部陨石的大约 8%。这可以说明它们起源于小行星带的内边缘,那里主要是由 Flora母体分裂产生的小行星家族。对此现象的一个可能的解释是,从主小行星带到地球轨道附近的物质运输,可能取决于其大小。

(据科学时报)