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Zero-set Quasi-convex Functions and the Optimality Conditions of Zero-set Quasi-convex Programming
零集拟凸函数和零集拟凸规划的最优性条件

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Abstract A new class of generalized convex functions, which are called zero-set quasi-convex functions are defined, and some of their basic properties are discussed. According to the properties of the functions, sufficient optimality conditions for the nonlinear zero-set quasi-convex programming with inequality constraints are given.

Key words convex function zero-set quasi-convex function, zero-set quasi-convex programming, optimality conditions

摘要:定义一种广义凸函数:零集拟凸函数,讨论其相关性质,并结合函数性质给出零集拟凸不等式约束规划的最优性条件.

关键词:凸函数 零集拟凸函数 零集拟凸规划 最优性条件
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Convexity plays a vital role in many aspects of mathematical programming, for example, sufficient optimality conditions and duality theorems. Over the years, many generalized convexities were presented[1~4]. In this paper, we introduce a new class of functions, which are called zero-set quasi-convex functions and present some results of them. The results of optimality in zero-set quasi-convex programming problems with inequality constrains are established.

In the following, we review several concepts of generalized convexity which have some relationships with zero-set quasi-convexity. In this paper, we assume that the set $S\subseteq R^n$ is a nonempty convex set.

1 Definitions

Definition 1.1[2] A real function $f: S\subseteq R^n\to R$ is said to be quasi-convex function, if

$f(\lambda x+(1-\lambda)y)\leqslant \max\{f(x),f(y)\},\forall x,y\in S,\forall \lambda\in(0,1)$.

Definition 1.2[2] Let $f: S\subseteq R^n\to R$ be a differentiable function. f is said to be pseudo-convex function, if $\forall x,y\in S$ with $\nabla f(x)^T(y-x)\geqslant 0$ one can get $f(y)\geqslant f(x)$.

2 Zero-Set quasi-convex functions and their properties

In this section, we present the definition of zero-set quasi-convex function and discuss its main properties.

Definition 2.1 A function $f: S\to R$ is said to be zero-set quasi-convex on Z, if

$f(\lambda x+(1-\lambda)y)\leqslant \max\{f(x),f(y)\},\forall x\in$

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$Z, \forall y \in S, \forall \lambda \in [0,1]$,
where $Z \subseteq \{x \in S | f(x) = 0\}$.

Proposition 2.1 If $f(x)$ is a quasi-convex function and $Z = \{x | f(x) = 0\} \neq \emptyset$, then $f(x)$ is a zero-set quasi-convex function on Z.

Remark 2.1 The converse of Proposition 2.1 is not necessarily true. A counterexample is given as follows.

Example 2.1 Let f: R→ R be defined as

$$f(x) = \begin{cases} (x+1)^2 - 2, x \leqslant 0, \\ (x-1)^2 - 2, x > 0. \end{cases}$$

The graph of the function f is shown in Fig. 1

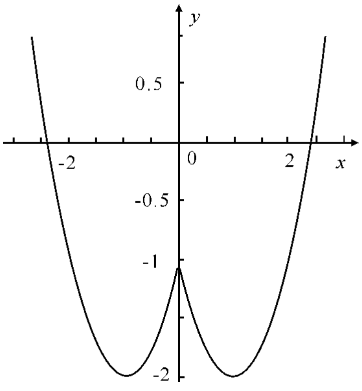

Fig. 1 Function f

Let $x_1 = -1, x_2 = 1, \lambda_0 = \dfrac{1}{2}$, then

$f(\lambda_0 x_1 + (1-\lambda_0)x_2) = f(0) = -1 > \max\{f(x_1), f(x_2)\} = -2$.

So, f is not a quasi-convex function. On the other hand, from Fig. 1, one can see that f is a zero-set quasi-convex function (one can also get from Theorem 2.2).

Proposition 2.2 Let f_i: S→ R$(i = 1,2,\cdots,m)$ be zero-set quasi-convex on $Z_i = \{x | f_i(x) = 0\} (i = 1,2,\cdots,m)$ and $Z = \bigcap_{i=1}^m Z_i \neq \emptyset$, then $f(x) = \max\{f_i(x), i = 1,2,\cdots,m\}$ is a zero-set quasi-convex on Z i.e.,

$f(\lambda x + (1-\lambda)y) \leqslant \max\{f(x), f(y)\}, \forall x \in Z, \forall y \in S, \lambda \in [0,1]$,

where $S = \bigcap_{i=1}^m S_i$.

Proof For $\forall x \in Z, \forall y \in S$ and $\forall \lambda \in [0,1]$, from the zero-set quasi-convexity of $f_i(x)$, we have

$f(\lambda x + (1-\lambda)y) = \max\{f_i(\lambda x + (1-\lambda)y), i = 1,2,\cdots,m\} \leqslant \max\{\max\{f_i(x), f_i(y)\}, i = 1,2,\cdots,m\} = \max\{\max\{f_i(x), i = 1,2,\cdots,m\}, \max\{f_i(y), i = 1,2,\cdots,m\}\} = \max\{f(x), f(y)\}$.

So, $f(x)$ is zero-set quasi-convex on Z.

Theorem 2.1 If continue function $f(x)$ is a zero-set quasi-convex function on $Z = \{x | f(x) = 0\}$, then $S_0 = \{x | f(x) \leqslant 0\}$ is a convex set.

Proof Suppose S_0 is not a convex set, then there exist $x, y \in S_0, \bar{\lambda} \in (0,1)$, such that $f(\bar{\lambda}x + (1-\bar{\lambda})y) > 0$. If $f(x) = 0$ or $f(y) = 0$, then from the zero-set quasi-convexity of $f(x)$ we have $f(\bar{\lambda}x + (1-\bar{\lambda})y) \leqslant \max\{f(x), f(y)\} \leqslant 0$. So $f(x), f(y) < 0$. By the continue property of $f(x)$, one knows that there exist $\underline{\lambda} \in (0, \bar{\lambda})$, $\bar{\bar{\lambda}} \in (\bar{\lambda}, 1)$, such that

$f(\underline{\lambda}x + (1-\underline{\lambda})y) = 0, f(\bar{\bar{\lambda}}x + (1-\bar{\bar{\lambda}})y) = 0$.

Let $\tilde{x} = \underline{\lambda}x + (1-\underline{\lambda})y, \tilde{y} = \bar{\bar{\lambda}}x + (1-\bar{\bar{\lambda}})y$, one can easily know that there exist $\lambda_0 \in (0,1)$ such that $\lambda_0 \tilde{x} + (1-\lambda_0)\tilde{y} = \bar{\lambda}x + (1-\bar{\lambda})y$.

On the other hand,

$f(\bar{\lambda}x + (1-\bar{\lambda})y) = f(\lambda_0 \tilde{x} + (1-\lambda_0)\tilde{y}) \leqslant \max\{f(\tilde{x}), f(\tilde{y})\}$.

This is a contradiction. So, $S_0 = \{x | f(x) \leqslant 0\}$ is a convex set.

Theorem 2.2 Let f: S→ R. If $S_\tau = \{x | f(x) \leqslant \tau, x \in S\}$ is a convex set for each $\tau \geqslant 0$ and $Z = \{x | f(x) = 0\} \neq \emptyset$, then $f(x)$ is a zero-set quasi-convex function on Z.

Proof For $\forall x \in Z, \forall y \in S$. Let $\tau_0 = \max\{f(x), f(y)\} \geqslant 0$, then $x, y \in S_{\tau_0}$. So, from the convexity of S_{τ_0}, we have $\lambda x + (1-\lambda)y \in S_{\tau_0}, \forall \lambda \in [0,1]$, i.e., $f(\lambda x + (1-\lambda)y) \leqslant \tau_0 = \max\{f(x), f(y)\}, \forall \lambda \in [0,1]$. The proof of this theorem is completed.

Theorem 2.3 Let f: S→ R is zero-set quasi-convex on $Z = \{x | f(x) = 0\}$. If f is differentiable, then for $\forall x \in Z, \forall y \in S$, we have

(i) if $f(y) \leqslant 0$, then $\nabla f(x)^T (y-x) \leqslant 0$;

(ii) if $f(y) \geqslant 0$, then $\nabla f(y)^T (x-y) \leqslant 0$;

(iii) if $\nabla f(x)^T (y-x) > 0$, then $f(y) > 0$;

(iv) if $\nabla f(y)^T (x-y) < 0$, then $f(y) < 0$.

Proof It is obviously that (i) and (iii), (ii) and (iv) are equivalent. It is only need to show the statements (i) and (ii).

(i) Suppose $x \in Z, y \in S$ and $f(y) \leqslant 0$, then,

$$\nabla f(x)^T (y-x) = f'(x; y-x) = \lim_{\lambda \to 0^+} \frac{f(x + \lambda(y-x)) - f(x)}{\lambda} \leqslant \lim_{\lambda \to 0^+} \frac{\max\{f(x), f(y)\} - f(x)}{\lambda} = 0.$$

The statement (i) holds.

(ii) The proof is similar with (i). The proof of this theorem is completed.

3 Optimality conditions

In this section, we apply the associated results to the nonlinear programming problem with inequality constraints as follows:

$$(P_g)\quad \begin{array}{ll} \min & f(x) \\ \text{s. t.} & g_i(x) \leqslant 0, i \in I = \{1, 2, \cdots, m\}, \\ & x \in R^n. \end{array}$$

Denote the feasible set of (P_g) by $S_g = \{x \in R^n \mid g_i(x) \leqslant 0, i \in I\}$. For convenience of discussion, we always assume that f and g_i are all differentiable and S_g is a nonempty set in R^n.

Theorem 3.1 If $g_i(x)(i \in I)$ are zero-set quasi-convex functions, then the feasible set S_g of problem (P_g) is a convex set.

Proof Let $S_i = \{x \mid g_i(x) \leqslant 0\}, i \in I$, from Theorem 2.1, one knows that $S_i(i \in I)$ are all convex sets. So, $S_g = \bigcap_{i \in I} S_i$ is a convex set.

Theorem 3.2 Assume that x^* is a KKT point of (P_g), and the function $f(x)$ is differentiable and pseudo-convex, $g_i(x)(i \in I)$ are differentiable and zero-set quasi-convex, then x^* is an optimal solution of the problem (P_g).

Proof For any $x \in S_g$, we have $g_i(x) \leqslant 0 = g_i(x^*)$, $i \in I(x^*) = \{i \in I \mid g_i(x^*) = 0\}$. Therefore, from the zero-set quasi-convexity of $g_i(x)$, $x \in S_g$ and Theorem 2.3, one can obtain $\nabla g_i(x^*)^T(x - x^*) \leqslant 0$ for $i \in I(x^*)$.

Since x^* is the KKT point of (P_g), there exist multipliers $u_i \geqslant 0$ such that

$$\nabla f(x^*) + \sum_{i \in I} u_i \nabla g_i(x^*) = 0, u_i g_i(x^*) = 0.$$

From the above equation, we have

$$\nabla f(x^*)^T(x - x^*) = -\sum_{i \in I} u_i \nabla g_i(x^*)^T(x - x^*) = -\sum_{i \in I(x^*)} u_i \nabla g_i(x^*)^T(x - x^*) \geqslant 0.$$

Hence from the pseudo-convexity of $f(x)$, one can conclude $f(x) \geqslant f(x^*)$. Therefore x^* is an optimal solution of the problem (P_g).

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