DOI: 10.13656/j.cnki.gxkx.2008.03.021
广西科学 Guangxi Sciences 2008, 15(3): 274～277, 281

Pricing Contingent Claims Attainable with Positive Probability*

依概率可达未定权益的定价

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Abstract Based on statistical analogies, the α- price of an American contingent claim is obtained, and a minimal hedge strategy is constructed. Enlightened by hedging the contingent claim in probability, we get the pricing formula of the contingent claim attainable in probability. At last we consider an example which is an one period model attainable in probability and get the fair price by this pricing formula, its special case is a non-attainable contingent claim.

Key words contingent claim, pricing, equity price, stopping time

摘要: 应用统计方法得出美式未定权益的 α 价格,并构造出它的一个最小套期保值策略,再根据美式未定权益的 α-价格推导出依概率可达未定权益的公平价格公式,并用该公式求出一个单时段市场上依概率可达未定权益的公平价格.

关键词: 未定权益 定价 公平价格 停时
中图法分类号: O213, F830.6 文献标识码: A 文章编号: 1005-9164(2008)03-0274-04

Since F Black and M Scholes[1] derived the formula of pricing options, the studies of pricing contingent claims in complete market had made a great progress. J C Cox, S A Ross and M Rubinstein[2] found a simple approach to pricing options. Follmer[3], Schweizer[3] and Nguyen[4] had made a profound study on the pricing of European contingent claim in incomplete markets. Youn[5] showed an example of non-attainable contingent claim in one period model. Altug[6] also showed some examples of complete and incomplete market models. Merton[7] studied a stock price evolve in both geometric Brownian motion and Poission process with varying jumps, and he showed that it is impossible to construct a hedging portfolio of stock and option in the particular market as Black and Scholes did for geometric Brownian motion. Naik and Lee[8] showed an example of contingent claim that can′t be replicated when the stock price follows a mixture process, and they used Merton′s model[7] to value a non-attainable contingent claim. Kallsen and Kühn[9] derived the American contingent claim price in incomplete market and explicitly explained the pricing derivatives of American and game type in incomplete markets.

In contrast, there are little corresponding literatures dealing with the contingent claims attainable in probability. Mel′nikov[10] brought a new idea for pricing and hedge European contingent claims with a positive probability. Actually, most of contingent claims attainable in probability are American. How to price and hedge these claims is becoming more and more urgent.

In this paper, the approach presented below is based on statistical analogies and assumes that the

收稿日期: 2007-05-08
修回日期: 2008-01-23
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* 湖南省自然科学基金项目(04JJ3076); 中南大学文理基金项目(0502011)资助。

American contingent claim attainable in probability. Just like Mel′nikov[10], it turns out that a satisfactory solution of the above problem is possible. Moreover, according to the α-price of American contingent claim, we get the fair price of a contingent claim attainable in probability. At the end, we will consider an example to explain the theory we get.

1 Some definitions

What we discuss below are based on Mel′nikov′s complete (B, S)-market[10] with deterministic $r_n > -1$ and $B_0 = 1$. We have given an American contingent claim (f, N), such that $E^* f_\tau > 0$ for each stopping time τ, where E^* is the average with respect to the martingale measure P^*. With the given contingent claim, the corresponding class of self-financing strategies is

$$SF(f, N) = \{\pi \in SF: X_\tau^\pi \geqslant f_\tau - E^* f_\tau\}.$$

Definition 1.1 For an American contingent claim $f = (f_n)_{n \leqslant N}$, set some levels of significance $\alpha \in (0, 1)$ of the contingent claim (f, N). We call a strategy $\pi \in SF(f, N)$ is an α-(x, f, N)-hedge if for $P' = P$ and $P' = P^*$,

$$P'\{X_\tau^\pi \geqslant f_\tau\} \geqslant 1 - \alpha, \tag{1.1}$$

over all stopping time τ.

The set of α-hedges is denoted by $\prod(x, f, N, \alpha)$.

Definition 1.2 A strategy $\pi \in SF(f, N)$ is an α-minimal hedge if there is a stopping time τ, such that

$$P'(X_\tau^\pi(\omega) = f_\tau(\omega)) = 1 - \alpha,$$

for all $\omega \in \Omega$.

Definition 1.3 Suppose (f, N) is an American contingent claim with $f = (f_n)_{n \leqslant N}$, we call $C(N, \alpha)$ is an α-price of the contingent claim (f, N) where

$$C(N, \alpha) = \inf\{x > 0 \mid \prod(x, f, N, \alpha) \neq \varnothing\}. \tag{1.2}$$

This is only a definition of α-price of an American contingent claim. By this definition we can′t calculate the concrete value nearly.

2 Main conclusion

Theorem 2.1 The α-price of an American contingent claim (f, N) is $(1 - \alpha)C$, where C is the fair price of the American contingent claim.

Proof Firstly, we establish that for any portfolio $\pi \in SF(f, N)$,

$$P^*\{X_{\tau^*}^\pi \geqslant f_{\tau^*}\} \leqslant \frac{X_0^\pi}{C}, \tag{1.3}$$

where τ^* is the optimal stopping time and $C = E^* \mathcal{X}^{-1}(U) f_{\tau^*}$ is the fair price of the American contingent claim.

Indeed, by Chebyshev′s inequality, the definition of class $SF(f, N)$, the martingale property of the measure P^* and the Mel′nikov′s formula, we can conclude formulae(1.3).

$$P^*\{X_{\tau^*}^\pi \geqslant f_{\tau^*}\} = P^*\{X_{\tau^*}^\pi - f_{\tau^*} + E^* f_{\tau^*} \geqslant E^* f_{\tau^*}\} \leqslant \frac{E^*(X_{\tau^*}^\pi - f_{\tau^*} + E^* f_{\tau^*})}{E^* f_{\tau^*}} = \frac{E^* X_{\tau^*}^\pi}{E^* f_{\tau^*}} = \frac{E^* E_{\tau^*}^{-1}(u) X_{\tau^*}^\pi}{E^* E_{\tau^*}^{-1}(u) f_{\tau^*}} = \frac{X_0^\pi}{C}.$$

From formulae (1.1) and formulae(1.3), the necessary condition for a strategy π in the class $SF(f, N)$ to be an α-(x, f, N)-hedge is

$$1 - \alpha \leqslant P^*\{X_{\tau^*}^\pi \geqslant f_{\tau^*}\} \leqslant \frac{x}{C}, \text{ or}$$

$$x \geqslant (1 - \alpha)C.$$

Then, we′ll construct a strategy π_α with initial value $(1 - \alpha)C$. If we can prove $\pi_\alpha \in \prod((1 - \alpha)C, f, N, \alpha)$ and there is an α-minimal hedge in the set of α-hedges $\prod((1 - \alpha)C, f, N, \alpha)$, then theorem 2.1 is tenable.

For the original measure P and martingale measure P^* we define the density is

$$Z_{\tau^*} = \frac{dP}{dP^*}. \tag{1.4}$$

It is clear that there is a $\lambda = \lambda(\alpha)$ such that

$$P^*\{Z_{\tau^*} \geqslant \lambda\} = 1 - \alpha. \tag{1.5}$$

With the aim of simplifying the arguments we assume that $\lambda \geqslant 1$.

Considering the martingales $M_n^\alpha = E^*(I_{\{Z_{\tau^*} < \lambda(\alpha)\}} \mid F_n)$ and $M_n^C = E^*(E_{\tau^*}^{-1}(U) f_{\tau^*} \mid F_n)$.

Since the market is complete, admit representations

$$M_n^\alpha = \alpha + \sum_{k=1}^n h_k E_k^{-1}(U) S_{k-1}(d_k - r_k),$$

$$M_n^C = C + \sum_{k=1}^n V_k \mathcal{X}^{-1}(U) S_{k-1}(d_k - r_k),$$

with predictable sequences (h_k) and (V_k).

For the initial value $X_0 = (1 - \alpha)C$, we define the portfolio $\pi_\alpha = (U_n^\alpha, V_n^\alpha)_{n \leqslant N}$ by the following formulas

$$V_n^\alpha = V_n - h_n C, \quad U_n^\alpha = \frac{X_{n-1}^{\pi_\alpha} - V_n^\alpha S_{n-1}}{E_{n-1}(U)}. \tag{1.6}$$

We will show that the so-called portfolio π_α is an α-((1

$-$ T$)C, f, N)$ $-$hedge.

Indeed, from reference[10] and formulae(1. 6), we have

$$E_n^{-1}(U)X_n^{c_T} = X_0 + \sum_{k=1}^{n} E_k^{-1}(U)V_k^T S_{k-1}(d_k - r_k) =$$

$$(1-T)C + \sum_{k=1}^{n} E_k^{-1}(U)(V_k^* - h_k C)S_{k-1}(d_k - r_k) =$$

$$(1-T)C + \sum_{k=1}^{n} E_k^{-1}(U)S_{k-1}V_k^*(d_k - r_k) -$$

$$\sum_{k=1}^{n} E_k^{-1}(U)S_{k-1}h_k C(d_k - r_k) = (1-T)C - (1-$$

$$T)C + M_n^C - C(M_n^T - T) . \qquad (1. 7)$$

From formulae(1. 7), we have

$$E_f^{-1}(U)X_f^{c_T}, E_f^{-1}(U)f_f - CI_{\{Z_f < \lambda\}} .$$

Therefore $E_f^{-1}(U)X_f^{c_T} \geqslant E_f^{-1}(U)f_f - C$.

The inequality above means that $c_T \in SF(f, N)$.

Further, it follows from formulae(1. 4) that

$$P^* \{X_f^{c_T} \geqslant f_f\} = P^* \{f_f - CI_{\{Z_f < \lambda\}}E_f(U) \geqslant f_f\} = P^* \{I_{\{Z_f < \lambda\}} \leqslant 0\} = P^* \{Z_f \geqslant \lambda\} = 1 - T. \qquad (1. 8)$$

Finally, from formulae(1. 4), formulae(1. 5) and formulae(1. 7)

$$P\{X_f^{c_T} \geqslant f_f\} = E^* I_{\{X_f^T \geqslant f_f\}} Z_f \geqslant E^* I_{\{X_f^T \geqslant f_f\}} I_{\{Z_f \geqslant \lambda\}} Z_f \geqslant \lambda(1-T) \geqslant 1 - T. \qquad (1. 9)$$

The relations between formulae (1. 8) and formulae(1. 9) show that the condition (1. 1) holds for the strategy c_T and hence c_T is an T$-((1-T)C, f, N)$ $-$hedge.

With the initial value

$$x = X_0^{c_T} = \sup_{\theta \leqslant f \leqslant N} E^* X_f^{-1}(U) f_f I_{\{X_f^{c_T} \geqslant f_f\}} = (1 - T) \sup_{\theta \leqslant f \leqslant N} E^* X_f^{-1}(U) f_f.$$

The discounted value of strategy c_T is

$$M_n^{c_T} = X_n^{c_T} / B_n.$$

The form of its representation is

$$M_n^{c_T} = M_0^{c_T} + \sum_{k=1}^{n} \frac{V_k^* S_{k-1}}{B_k}(d_k - r_k), \qquad (1. 10)$$

where, $V_k^* = V_k B_k S_{k-1}^{-1} \in F_{k-1}$, (V_k) is a predictable sequence.

Let $Y_n = \sup_{n \leqslant f \leqslant N} E^* (\frac{f_f I_{\{X_f^c \geqslant f_f\}}}{B_f} | F_n)$.

It is clear that Y_n is a supermartingale with respect to P^* . According to the Doob decomposition for supermartingales, we have

$$Y_n = M_n - A_n(P^* - a.s.),$$

where $M = (M_n)_{n \leqslant N}(M_0 = Y_0)$ is a martingale, and $A = (A_n)_{n \leqslant N}(A_0 = 0)$ is a predictable nondecreasing stochastic sequence.

Since (B, S)$-$market[10] is complete, admit representation

$$M_n = M_0 + \sum_{k=1}^{n} \frac{V_k^* S_{k-1}}{B_k}(d_k - r_k) . \qquad (1. 11)$$

Compare formulae (1. 10) and formulae(1. 11), we conclude that

$$M_n^{c_T} = M_n, n \leqslant N.$$

Consequently

$$X_n^{c_T} = M_n^{c_T} B_n = M_n B_n = (Y_n + A_n)B_n \geqslant Y_n B_n =$$

$$\sup_{n \leqslant f \leqslant N} E^* (\frac{f_f I_{\{X_f^{c_T} \geqslant f_f\}}}{B_f} | F_n\}B_n =$$

$$\sup_{n \leqslant f \leqslant N} E^* (X_f^{-1}(U)X_n(U) f_f I_{\{X_f^c \geqslant f_f\}} | F_n) \geqslant f_n I_{\{X_n^c \geqslant f_n\}}. \qquad (1. 12)$$

It follows from formulae(1. 12) that

$$P'\{X_n^{c_T} \geqslant f_n\} = 1 - T,$$

for $n = 0, 1, \cdots, N$, consequently $c_T \in \prod(x, f, N, T)$.

From the definition of Y_n, in the stochastic interval $[0, f]$ there is

$$Y_n(k) = E^* (Y_{n+1} | F_n), A_n(k) = 0.$$

From formulae(1. 12) it follows directly that in the same interval $[0, f]$ there is

$$X_n^{c_T}(k) = \sup_{n \leqslant f \leqslant N} E^* (X_f^{-1}(U)X_n(U) f_f I_{\{X_f^c \geqslant f_f\}} | F_n).$$

Considering the definition of f, the property of Y_n in $[0, f]$, and $A_n = \sum_{k=1}^{n}(Y_{k-1} - E^* (Y_k | F_{k-1}))$, we know $A_f(k) = 0$. So

$$P'\{X_f^{c_T}(k) = Y_f(k)B_f(k) = f_f(k)\} = 1 - T, \qquad (1. 13)$$

and c_T is an T$-$minimal hedge strategy.

Corollary 2. 1 Consider a complete (B, S)$-$market, if we set the risk sustaining level $T \in (0, 1)$, then we can hedge the contingent claim by initial assets $(1-T)C$, where C is the initial funds of hedge contingent claim with probability 1.

Remark 2. 1 The contingent claim can be European or American. The case of European has been proved by Mel′nikov, the case of American have been proved in this paper above.

What have been obtained show that it is possible to hedge a contingent claim with a specified probability $1 - T$. Further, the initial value can be reduced by the amount of TC, though the accepted contingent claim with a risk T can not be repaid.

Theorem 2.2 The fair price V of a contingent claim attainable with probability $1-$ T is

$$V = \frac{1}{1+r}[(1-T)C' + TV'] = \frac{E'f}{1+r},$$

where $C' = (1+r)C, V' = (1+r)V$, C is the fair price of the contingent claim when it is attainable with probability 1, and V is the neutral price of the contingent claim when it is non-attainable, E' is the average with respect to measure P'.

Proof Let C is the fair price of the contingent claim when it is attainable and V is the neutral price of the contingent claim when it is non-attainable, herein the contingent claim is attainable with probability $1-$ T, namely, non-attainable with probability T. So the fair price V is the average of C and V with respect to measure P'.

If we denote

$$C' = (1+r)C \text{ and } V' = (1+r)V,$$

then V can be writen by

$$V = \frac{1}{1+r}[(1-T)C' + TV'] = \frac{1}{1+r}E'f.$$

Example 2.1 We consider an one period model in which a stock take three possible values with a positive probability T. u and d are both regular movements with probability of p_1 and p_2 respectively, and j is an unexpected rare jump with a small probability p_3. In this case, the contingent claim can not be replicated, and the stock can also take two possible values with probability $1-$ T. u' and d' are both regular movements with probability of p_1' and p_2' respectively, and without jump. In this case, the contingent claim can be replicated. So, the contingent claim is attainable in probability.

By theorem 2.2, let

$$P^{*'} = \frac{1+r-d'}{u'-d'}.$$

It is clearly that the fair price of the contingent claim is

$$C = \frac{1}{1+r}[a'_1P^{*'} + a'_2(1-P^{*'})].$$

The neutral price of the contingent claim by reference [7] is

$$V = \frac{1}{1+r}[a_1P^* + a_2(1-P^*)](1-p_3) + \frac{1}{1+r}a_3p_3,$$

where $P^* = \frac{1+r-d}{u-d}$. So the fair price of the contingent claim is

$$V = \frac{1-T}{1+r}[a_1'P^{*'} + a_2'(1-P^{*'})] + \frac{T}{1+r}\{[a_1P^* + a_2(1-P^*)](1-p_3) + a_3p_3\} = \frac{1-T}{1+r}C' + \frac{T}{1+r}V' = \frac{1}{1+r}E'f,$$

where $C' = [a_1'P^{*'} + a_2'(1-P^{*'})]$, $V' = [a_1P^* + a_2(1-P^*)](1-p_3) + a_3p_3$, and E' is the average with respect to P'.

Remark 2.2 In particularly, if $u' = u$, $d' = d$, and $P^{*'} = P^*$, then, from theorem 2.2

$$V = \frac{1}{1+r}[a_1P^* + a_2(1-P^*)](1-Tp_3) + \frac{1}{1+r}Ta_3p_3.$$

Youn[5] also considered an one period model in which a stock can take three possible values, u and d, both regular movements with probability of 0.59 and 0.39 respectively, and j is an unexpected rare jump with a small probability 0.02. A contingent claim (a_1, a_2, a_3) is not necessarily replicated by portfolio of the stock and the riskless bond. Since a contingent claim space is R^3, a stock value vector (u,d,j) and a bond value vector $(1+r, 1+r, 1+r)$ can not span the contingent claim space, and Youn used the Merton formula to calculated the value of the one period conditioned on jump:

$$\{\frac{1}{1+r}[a_1p^* + a_2(1-p^*)]\}(1-p_3) + \frac{1}{1+r}p_3.$$

Therefore, in this case, a contingent claim attainable with probability $1-$ T is equivalent to a non-attainable contingent claim with a jump with probability Tp_3.

References

[1] Black F, Scholes M. The pricing of options and corporate liabilities[J]. Journal of Political Economy, 1973, 81: 637-654.

[2] Cox J C, Ross S A, M Rubinstein. Option pricing: a simplified approach[J]. Journal of Financial Economies, 1979(7): 229-268.

[3] Follmer H, Schweizer M. Hedge of contingent claims under incomplete information [J]. Applied Stochastic Analysis, 1990(4): 389-414.

[4] Nguyen X L. Hedge of contingent claims in incomplete markets[J]. Journal of Project Report, 2002(2): 324-339.

[5] Hyungho Youn. The various valuation methods for non-attainable contingent claim[J]. The Journal of Finance, 2004(1): 121-149.

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$$Z_j)\Delta x_i(k)\|_2^2 = \sum_{i=1}^{m_1}|\Delta x_i(k)\sum_{j=1}^{m_2} f'(a_i, Z_j)\Delta y_j(k)|^2 + \sum_{j=1}^{m_2}|\Delta y_j(k)\sum_{i=1}^{m_1} f'(a_i, Z_j)\Delta x_i(k)|^2.$$

即学习率 u 为

$$0 < u < 2/\left[\sum_{i=1}^{m_1}|\Delta x_i(k)\sum_{j=1}^{m_2} f'(a_i, Z_j)\Delta y_j(k)|^2 + \sum_{j=1}^{m_2}|\Delta y_j(k)\sum_{i=1}^{m_1} f'(a_i, Z_j)\Delta x_i(k)|^2\right]$$

时,有 $\Delta V(k) < 0$,从而算法是收敛的.证明完毕.

4 数值积分算例

例 求积分 $\int_{1.4}^{2.0}\int_{1.0}^{1.5}\ln(x+2y)\,dy\,dx$.

在 MATLAB语言环境下,取神经网络结构为 $(40+50)\times 2000\times 1$(即 $m_1=40, m_2=50$),性能指标 $J=10^{-16}$,在区域 $[1.4, 2.0]\times[1.0, 1.5]$上有 50 个分法,训练样本集为 $\{((\Delta X_k, \Delta Y_k), d) | k = 1, 2, \cdots, 50\}$,所得到结果为 0.429554526002843,迭代 38 次,而积分准确值为 0.42955452600,此结果与精确值相比精确到 2.843×10^{12}.性能指标 J 与迭代次数的变化曲线见图 2.

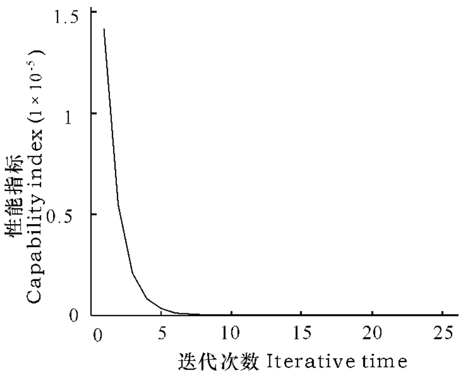

图 2 性能指标与迭代次数的变化曲线

Fig. 2 Curve of capability and iterative time

文献 [9]用复化辛卜生方法求解,所得结果为 0.4295524387,此计算值与精确值相比精确到 2.1×10^{-6}; 文献 [10]用高斯型求积公式计算,所得结果为 0.4295545313,此结果与精确值相比精确到 4.8×10^{-9}.

与文献 [9]复化辛卜生公式和文献 [10]的高斯型求积公式相比,本文提出的算法具有计算精度高收敛速度快的特点.

参考文献:

[1] Carlet P G. The finite element method for elliptie problems[M]. North-Holland: Amsterdam, 1978.

[2] 徐利治,周蕴时.高维数值积分 [M].北京:科学出版社, 1980.

[3] Davis P J, Rabinow P J. Method of numerical integration blaaisdell[M]. Waltham: Mass, 1975.

[4] 王能超.数值分析简明教程 [M].北京:高等教育出版社, 1997: 66-296.

[5] 李岳生,黄友谦.数值逼近 [M].北京:人民教育出版社, 1978: 76-204.

[6] 刘玉琏,傅沛仁.数学分析讲义:下册 [M].北京:高等教育出版社, 2001: 309-342.

[7] 翁显炎.也谈黎曼积分的定义 [J].丽水师范专科学校学报, 1989, 1: 25-29.

[8] 张永锋.关于重积分定义的注记 [J].安康师专学报, 1995(1): 62-66.

[9]《现代应用数学手册》编委会.计算与数值分析卷 [M].北京:清华大学出版社, 2001: 232-239.

[10] Curtis F Gerald (美), Patrick O Wheatley.应用数值分析 [M].吕淑娟,译.北京:机械工业出版社, 2003: 249-253.

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[6] Sumru G Altug. Complete and incomplete market models [J]. Koc University and CEPR, 2006(3): 213-303.

[7] Merton R C. Option pricing when underlying stock returns are discontinuous [J]. Journal of Financial Economics, 1976(3): 125-144.

[8] Naik V, Lee M. General equailibrium pricing of options on the market portfolio with discontinuous returns [J]. The Review of Financial Studies, 1990(3): 493-521.

[9] Kallsen Jan, Kuhn Christoph. Pricing derivatives of American and game type in incomplete markets [J]. Finance and Stochastics, 2004(8): 261-284.

[10] Mel'nikov A V. Financial markets stochastic analysis and the pricing of derivative securities [M]. New York: American Mathematical Society, 1999: 21-53.

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