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On s-Completion of Maximal Subgroups of Finite Groups*

有限群的极大子群 s-完备

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Abstract Let G be a group, denote $M_H(G)=\{M \mid M$ is a maximal subgroup of G such that $H \ntrianglelefteq M\}$ where H is a given normal subgroup of G. We investigated the properties of s-completion by the set $M_H(G)$, and obtained some new conditions for the solvability of finite groups.

Key words finite groups, solvable group, s-completion

摘要: 利用集合 $M_H(G)=\{M \mid M$是不包含 H的 G之极大子群$\}$研究有限群 G的极大子群 s完备的性质,得到有限群 G可解的一些新判据.

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Given a maximal subgroup M of group G, a completion C of M in G is a subgroup such that $C \nsubseteq M$ while $H \subseteq M$ whenever $H < C$ and $H \trianglelefteq G$. A completion C of M is called maximal if M has no any completion which contains C properly. $K(C)$ denotes the group generated by all proper subgroups of C which are normal in G, then $K(C) < C$ and $K(C) \trianglelefteq G$.

In reference [1], Deskins introduced the concept of completions for a maximal subgroup of a finite group. In reference [2], Deskins showed that a group G is solvable if and only if for every maximal subgroup M of G has a maximal completion C such that $C/K(C)$ is nilpotent with Sylow 2-subgroups of class at most 2. Deskins conjectured that a group G is supersolvable if and only if every maximal subgroup M of G has a maximal completion C such that $CM=G$ and $C/K(C)$ is cyclic. In fact, Bollester-Bolinches and Ezquerro[3] pointed out that the conjecture is false. Later, Zhao[4] proved that the group which satisfies the conditions in Deskins′ conjecture is supersolvable or has a homomorphic image isomorphic to S_4. In reference [5], Li got a complete characterization of supersolvable groups by means of maximal completions. In reference [6], Li and Zhao have further weakened the condition of maximal completion by defining s-completions.

In this paper, we investigated the properties of s-completion by the set $M_H(G)$, and obtained some new conditions for the solvability of finite groups, where $M_H(G)=\{M \mid M$ is a maximal subgroup of G such that $H \ntrianglelefteq M\}$.

Throughout this paper, all groups are finite groups. Our terminologies and notations are

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standard, see reference [7] and reference [8].

1 Definitions and lemmas

Definition 1.1[6] Given a maximal subgroup M of group G, a completion C of M is called an s-completion if either $C=G$ or there exists a subgroup D of G, which is not a completion of M, such that D contains C as a maximal subgroup.

A maximal completion must be an s-completion. The examples given in reference [6] show that the converse is not true in general.

Example 1.1 Take $G=\mathrm{Aut}(\mathrm{PSL}(2,25))=[\mathrm{PGL}(2,25)]Z_2$, the semidirect product of PGL(2, 25) by the cyclic group Z_2 of order 2[8]. Write $G_1=\mathrm{PGL}(2,25)$ and $G_2=\mathrm{PSL}(2,25)$. Then G has a unique chief series $G>G_1>G_2>1$. The group G_2 has maximal subgroups D_{24} and D_{26}, the dihedral groups of order 24 and 26, respectively. Furthermore, G has a maximal subgroup $M=N_G(D_{24})$. Take $C=N_G(D_{26})$, a maximal subgroup of G_1. Since C has no non-trivial G-invariant subgroup and C is not contained in M, it is a completion of M. Furthermore, C is an s-completion of M because G_1 is not a completion of M. We see that C is not a maximal completion of M since $N_G(D_{26})$ is also a completion of M and contains C properly.

Example 1.2 Let $G=S_4\times Z_2$. Take $M=S_3\times Z_2$, which is a maximal subgroup of G, and take a cyclic subgroup C of G with order 4 contained in S_4, then C is an s-completion of M but not a maximal completion of M.

Definition 1.2 $M_H(G)=\{M \mid M$ is a maximal subgroup of G such that $H\not\leq M\}$, where H is a given normal subgroup of G. $D(G)=\{M<\cdot G \mid |G:M|$ is a composite number$\}$.

Lemma 1.1[6] Let F be a formation and G be a group. If $G\notin F$ then there exists a normal subgroup N of G such that $G/N\in b(F)$, the Q-boundary of F, i.e., $G/N\notin F$ but every proper homomorphic image of G/N belongs to F. Furthermore, G/N has a unique minimal normal subgroup.

Lemma 1.2[6] Let G be a group and M be a maximal subgroup of G. Assume that N is a normal subgroup of G contained in M such that G/N has a unique minimal normal subgroup U/N with $U\not\subseteq M$. Furthermore, assume that C is an s-completion of M such that $C/K(C)\in F$, where F is a subgroup-closed homomorph, but $U/N\notin F$. Write $C^*=NC$. Then C^* is an s-completion of M in G satisfying

(1) $C^*\in K(C^*)$ in F and $N=k(C^*)$;

(2) C^* is a maximal subgroup of the group C^*U.

2 Main results

Theorem 2.1 Suppose G is a finite group, H is a normal subgroup of G. If for every non-nilpotent maximal subgroup $M\in M_H(G)\cap D(G)$, there exists an s-completion C of M such that $C/K(C)$ is nilpotent with Sylow 2-subgroups of class at most 2, then H is solvable.

Proof Assume the result is not true, and let G be a counterexample. Since the class of all solvable groups is a saturated formation, by lemma 1.1, there exists a normal subgroup N of G such that G/N has a unique minimal normal subgroup U/N (so $U/N\leq H/N$), which is insolvable. Then U/N is a non-abelian characteristically simple group. In particular, U/N has no non-trivial normal p-subgroup for any prime p. We claim that G has a non-nilpotent maximal subgroup M of composite index such that $N\subseteq M$, but $U\not\subseteq M$. For this, let q be the largest prime factor dividing $|U/N|$, and $Q/N\in \mathrm{Syl}_q(U/N)$, then Q/N is not normal in G/N and hence $N_{G/N}(Q/N)<G/N$. So there exists a maximal subgroup of G/N denote M/N such that $N_{G/N}(Q/N)<M/N$. This implies that M contains $N_G(Z(J(Q)))$ and N. By the Frattini argument, $G=N_G(Z(J(Q)))U=MU$, so $U\not\subseteq M$. Observe that $|G:M|=|U:(M\cap U)|\equiv 1(\mathrm{mod}\,q)$, so $|G:M|$ must be composite. If M is nilpotent then, as a subgroup of M, $N_U(Z(J(Q)))$ is also nilpotent. Note that q is odd, the Glaubermen-Thompson Theorem asserts that U is q-nilpotent, contrary to the fact that U/N is a non-abelian characteristically simple group. Hence, $M\in D(G)$. Clearly, $H\not\leq M$, so $M\in M_H(G)\cap D(G)$. By the hypothesis, M has an s-completion C such that $C/k(C)$ is nilpotent with Sylow 2-subgroups of class at most 2. Of course, the class of all nilpotent groups

with Sylow 2-subgroups of class at most 2 is subgroup-closed homomorph, and U/N does not belong to this class. By lemma 1.2, there exists an s-completion C of M such that $N=K(C)$ and C is a maximal subgroup of UC. Now $UC/N=U/N\cdot C/N$ and $C/N=C/K(C)$ is a nilpotent maximal subgroup of UC/N with sylow 2-subgroups of class at most 2. By the Deskins-Janko-Thomopson Theorem[7], UC/N must be solvable, so U/N is solvable, which is a contradiction. The proof of the theorem 2.1 is now complete.

Corollary 2.1[6] A group G is solvable if and only if for every non-nilpotent maximal subgroup M of G of composite index, there exists an s-completion C of M such that $C/k(C)$ is nilpotent with Sylow 2-subgroups of class at most 2.

Proof Set $H=G$, so $M_H(G)\cap D(G)=D(G)$, by theorem 2.1, the sufficiency part is hold, and the necessity part of the corollary is obvious.

Corollary 2.2 A group G is solvable if and only if for every non-nilpotent maximal subgroup M of G of composite index, there exists a maximal completion C of M such that $C/K(C)$ is nilpotent, with Sylow 2-subgroups of class at most 2.

Proof Since a maximal completion is an s-completion, by corollary 2.1, the conclusion holds.

From the definition of Deskins´completions, we see that a completion of maximal subgroup M may be a conjugation of M.

Theorem 2.2 Suppose G is a finite group, H is a normal subgroup of G. If for every $M\in M_H(G)\cap D(G)$, there exists an s-completion C of M such that $C/K(C)$ is nilpotent and $C^x\nsubseteq M$ for any $x\in G$, then H is solvable.

Proof Assume the result is false and let group G be a counterexample. As in the proof of theorem 2.1, there exists a normal subgroup N of G such that G/N has a unique minimal normal subgroup U/N (so $U/N\leq H/N$) which is insolvable, and G has a maximal subgroup M of composite index such that $N\leq M$ but $U\nsubseteq M$. So $M\in M_H(G)\cap D(G)$. By lemma 1.2, we may choose an s-completion C of M such that $C/K(C)=C/N$ is nilpotent, $C^x\nsubseteq M$ for any $x\in G$, and C is a maximal subgroup of UC.

Consider the group $E/N=U/N\cdot C/N$. Since C/N is a nilpotent maximal subgroup of E/N, by a theorem of Rose[9], K/N is normal in E/N, where K/N is the normal 2-complement of C/N. But U/N has non-trivial solvable normal subgroup, so K/N must be the identity group. Consequently, C/N is a 2-group and hence be a Sylow 2-subgroup of E/N.

Write $T=C\cap U$, then $N\leq T$ and T/N is a Sylow 2-subgroup of U/N. By the Feit-Thompson Theorem on groups of odd order, $T/N\neq 1$ and T is non-normal in U. Applying the Frattini argument, we have $G=N_G(T)U=M^*U$, where M^* is a maximal subgroup of G containing $N_G(T)$. It is obviously that $|G:M^*|$ is an odd number and $C\leq M^*$, also $M^*\cap U \trianglelefteq M^*$ and $C\leq E\cap M^*$ but $E\nsubseteq M^*$. We see that $E\cap M^*=C$ since C is maximal in E. It follows that $C(U\cap M^*)=CU\cap M^*=E\cap M^*=C$, so $(M^*\cap U)/N$ is a 2-group. Therefore $|G:M^*|=|U:(M^*\cap U)|$ can not be a prime, otherwise U/N would be solvable by the Burnside (p,q)-Theorem, a contradiction.

Now M^* is a maximal subgroup of G with composite index and $N\subseteq M^*$ but $U\nsubseteq M^*$. By hypothesis, M^* has an s-completion C^* such that $C^*/K(C^*)$ is nilpotent and $(C^*)^x\nsubseteq M^*$ for any $x\in G$. Replacing M by M^*, we have $N=K(C^*)$ and C^*/N is a 2-subgroup of G/N. Since M^*/N contains a Sylow 2-subgroup of G/N, by the Sylow Theorem, there exists an element x in G such that $(C^*)^x\leq M^*$, which is final contradiction. Thus, the proof is complete.

Corollary 2.3[6] A group G is solvable if and only if for every maximal subgroup M of G with composite index, there exists an s-completion C of M such that $C/k(C)$ is nilpotent and $C^x\nsubseteq M$ for any $x\in G$.

Corollary 2.4 A group G is solvable if and only if for every maximal subgroup M of G with composite index, there exists a maximal completion C of M such that $C/K(C)$ is nilpotent and $C^x\nsubseteq M$ for any $x\in G$.

Theorem 2.3 Suppose G is a finite group, H is a normal subgroup of G. If for every normal maximal subgroup $M\in M_H(G)\cap D(G)$, there exists a normal s-completion C such that $C/K(C)$ is solvable, then

H is solvable.

Proof Assume the result is false and let group G be a counterexample. As in the proof of theorem 2.1, there exists a normal subgroup N of G such that G/N has a unique minimal normal subgroup U/N (so $U/N \leq H/N$) which is insolvable. Set q be the largest prime factor dividing $|U/N|$ and $Q/N \in Syl_q(U/N)$. So Q is not normal in U and we can choose a maximal subgroup M of G to contain $N_G(Q)$ and N. By the Frattini argument, $G = N_G(Q)U = MU$, so $U \nsubseteq M$. Observe that $|G:M| = |U:(M \cap U)| \equiv 1 \pmod{q}$, hence $|G:M|$ is composite. So $M \in M_H(G) \cap D(G)$, by the hypothesis, there exists a normal s-completion C of M such that $C/K(C)$ is solvable. By lemma 2.2, we may choose an s-completion $C^* = CN$ of M such that $C^*/k(C^*) = C^*/N$ is solvable, and C^* is a maximal subgroup of UC^*.

Consider the group $E/N = U/N \cdot C^*/N$. Since C^*/N is a normal solvable maximal subgroup of E/N, thus E/C is solvable, consequently U/N is solvable, a contradiction. So the proof is complete.

Corollary 2.5 Suppose G is a finite group, if for every normal maximal subgroup $M \in D(G)$, there exists a normal s-completion C such that $C/k(C)$ is solvable, then G is solvable.

Corollary 2.6 Suppose G is a finite group, if for every normal maximal subgroup $M \in D(G)$, there exists a normal maximal completion C such that $C/k(C)$ is solvable, then G is solvable.

References

[1] Deskin W E. On maximal subgroups[J]. Amer Math Soc, 1959, 1: 100-104.

[2] Deskin W E. A note on the index complex of a maximal subgroup[J]. Arch Math (Basel), 1990, 54: 236-240.

[3] Ballester-Bolincees A, Ezquerro L E. On the index complex of a maximal subgroup of a finite group[J]. Proc Amer Math Soc, 1992, 114: 325-330.

[4] Zhao Yaoqing. On the index complex of a maximal subgroup and the supersolvability of a finite group[J]. Comm Algebra, 1996, 24: 1785-1791.

[5] Li Shirong. The Deskins index complex and the supersolvabilibty of finite groups [J]. J pure Appl Algebra, 1999, 144: 297-302.

[6] Li Shirong, Zhao Yaoqing. On s-completions of maximal subgroups of finite groups [J]. Algebra Collogmum, 2004, 11: 411-420.

[7] Huppert B. Endliche gruppen 1[M]. Berlin: Springer, 1976.

[8] Gorenstein D. Finite group[M]. New York: Chelsea, 1980.

[9] Rose J S. On finite insoluble groups with nilpotent maximal subgroups[J]. J Algebra, 1977, 48: 182-196.

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参考文献:

[1] Deskins W E. On maximal subgroups[J]. Proc Symp Pure Math, 1959, 1: 100-104.

[2] Wang Yanming. C-normality of groups and its properties[J]. Journal of Algebra, 1996, 180: 954-965.

[3] Li Shirong, Wang Yanming. On C-section and C-index of finite groups[J]. Journal of Pure and Applied Algebra, 2000, 151: 309-319.

[4] 徐明曜,李慧陵,李世荣,等.有限群导引(上,下)[M].北京:科学出版社,1999.

[5] Petter M R. A note on finite groups having a fixed-point-free automorphism [J]. Proc Amer Math Soc, 1975, 52: 79-80.

[6] 王品超,杨兆兴.有限群极大子群的正规指数[J].工程数学学报,1994,11(1): 42-48.

[7] Zhao Yaoqing. On the deskins completions and theta completions for maximal subgroups [J]. Communications in Algebra, 2000, 28(1): 375-385.

[8] Gorenstein D. Finite groups[M]. New York: Chelsea, 1980.

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