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二元积分和不等式中未知函数估计式的证明*

Proof of Estimation of Unknown Function in a Class Integral-sum Inequalities in Two Variables

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摘要：利用数学归纳法和不等式技巧证明一个已知的积分和不等式中未知函数的估计式成立。

关键词：积分和不等式 未知函数 数学归纳法
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Abstract The estimation of unknown function in a class of integro-sum inequalities has been proved by inductive approach and methodology of inequalities.

Key words integro-sum inequalities, unknown function, mathematical induction

尽管多数脉冲微分方程无法求出精确解，但是我们可以利用积分不等式技巧对解的模进行估计。模的估计可以证实微分方程解的存在性、有界性、唯一性和稳定性等[1~3]。为了进一步研究积分和不等式中未知函数的估计，受到文献[1~4]的启发，本文对文献[2]中给出的一个积分和不等式中未知函数的估计进行详细地证明。

1 未知函数的估计式

令 $K = \bigcup_{k,j \geqslant 1} K_{kj} = \{(t,x): t \in [t_{k-1}, t_k), x \in [x_{j-1}, x_j)\}, k = 1,2,\cdots, j = 1,2,\cdots\}$，$G = \{(t_i, x_i), i = 1,2,\cdots\}$。

命题 1[2] 假设非负函数 $u(t,x)$ 在 $K \backslash G$ 内连续，在点集 G 上 $u(t_i - 0, x_i - 0) \neq u(t_i + 0, x_i + 0)$，并且满足积分和不等式

$$u(t,x) \leqslant a(t,x) + \int_{t_0}^{t}\int_{x_0}^{x} b(F,s)u(F,s)\,dF\,ds + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} U_i u^m(t_i - 0, x_i - 0), \quad (1)$$

其中 $0 < m < 1$，$t_0 \geqslant 0$，$x_0 \geqslant 0$，$a(t,x)$ 是 K 上的正函数，U_i 都是非负常数，$b(t,x) \geqslant 0$ 还满足条件：$b(t,x) = 0$，$\forall (t,x) \in K_{ij}, i \neq j, i,j = 1,2,\cdots$，$\lim_{i\to\infty} t_i = \infty$，$\lim_{i\to\infty} x_i = \infty$，$(t_i, x_i) < (t_{i+1}, x_{i+1})$ 指的是 $t_i < t_{i+1}$，$x_i < x_{i+1}$。那么积分和不等式中的未知函数 $u(t,x)$ 有估计式

$$u(t,x) \leqslant \tilde{a}(t,x) \prod_{(t_0,x_0)<(t_i,x_i)<(t,x)} (1 + U_i \tilde{a}^{m-1}(t_i,x_i)) \exp\left(\int_{t_0}^{t}\int_{x_0}^{x} b(F,s)\,dF\,ds\right), \quad (2)$$

其中，$\tilde{a}(t,x) = \max_{t_0<F<t, x_0<s<x} a(F,s)$。

2 估计式的证明

由 $\tilde{a}(t,x)$ 的定义，可以得出 $\tilde{a}(t,x) \geqslant a(t,x)$，并且知道 $\tilde{a}(t,x)$ 关于每个变量都是单调的，即

$$\forall p < P, q < Q \Rightarrow \tilde{a}(p,q) \leqslant \tilde{a}(P,Q).$$

定理 1 $u(t,x) \leqslant \tilde{a}(t,x) \prod_{(t_0,x_0)<(t_i,x_i)<(t,x)} (1 + U_i \tilde{a}^{m-1}(t_i,x_i)) \exp\left(\int_{t_0}^{t}\int_{x_0}^{x} b(F,s)\,dF\,ds\right)$，

其中，$\tilde{a}(t,x) = \max_{t_0<F<t, x_0<s<x} a(F,s)$。

证明 用归纳法证明。如果 $(t,x) \in K_{11}$，那么积分和不等式(1)变成

$$u(t,x) \leqslant a(t,x) + \int_{t_0}^{t}\int_{x_0}^{x} b(F,s)u(F,s)\,dF\,ds.$$

进一步有

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$$u(t,x)\leqslant \tilde{a}(t,x)+\int_{t_0}^{t}\int_{x_0}^{x} b(\mathrm{F},s)u(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s. \quad (3)$$

令 $H_1(t,x)=\tilde{a}(t,x)+\int_{t_0}^{t}\int_{x_0}^{x} b(\mathrm{F},s)u(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s$，则可以看出 $H_1(t,x)$ 关于每个变量都是单调的，并且 (3) 式等价于 $u(t,x)\leqslant H_1(t,x)$. 用微分法消去积分号内的未知函数，那么有

$$\frac{\frac{\partial}{\partial t}H_1(t,x)}{H_1(t,x)}\leqslant \frac{\frac{\partial}{\partial t}\tilde{a}(t,x)}{\tilde{a}(t,x)}+\frac{1}{H_1(t,x)}\int_{x_0}^{x} b(t,s)u(t,s)\,\mathrm{d}s\leqslant \frac{\frac{\partial}{\partial t}\tilde{a}(t,x)}{\tilde{a}(t,x)}+\frac{1}{H_1(t,x)}\int_{x_0}^{x} b(t,s)H_1(t,s)\,\mathrm{d}s\leqslant \frac{\frac{\partial}{\partial t}\tilde{a}(t,x)}{\tilde{a}(t,x)}+\int_{x_0}^{x} b(t,s)\mathrm{d}s. \quad (4)$$

不等式 (4) 两边从 t_0 到 t 积分得到

$$\ln H_1(t,x)\leqslant \ln\tilde{a}(t,x)+\int_{t_0}^{t}\int_{x_0}^{x} b(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s. \quad (5)$$

由 (5) 式推出

$$u(t,x)\leqslant \tilde{a}(t,x)\exp\left(\int_{t_0}^{t}\int_{x_0}^{x} b(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s\right). \quad (6)$$

如果 $(t,x)\in \mathrm{K}_{i1}\cup \mathrm{K}_{1j}, i,j=2,3,\cdots$，积分和不等式 (1) 变成

$$u(t,x)\leqslant a(t,x)+\int_{t_0}^{t_1}\int_{x_0}^{x} b(\mathrm{F},s)u(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s,$$

或

$$u(t,x)\leqslant a(t,x)+\int_{t_0}^{t}\int_{x_0}^{x_1} b(\mathrm{F},s)u(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s,$$

用类似的证明方法，可以得到形如 (2) 式的估计.

如果 $(t,x)\in \mathrm{K}_{22}$，则积分和不等式 (1) 变成

$$u(t,x)\leqslant a(t,x)+\int_{t_0}^{t}\int_{x_0}^{x} b(\mathrm{F},s)\cdot u(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s+\mathrm{U}u^m(t_1-0,x_1-0)\leqslant \tilde{a}(t,x)+\mathrm{U}\tilde{a}^m(t_1,x_1)\exp\left(\int_{t_0}^{t_1}\int_{x_0}^{x_1} b(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s\right)+\int_{t_0}^{t_1}\int_{x_0}^{x_1} b(\mathrm{F},s)\tilde{a}(\mathrm{F},s)\exp\left(\int_{t_0}^{\mathrm{F}}\int_{x_0}^{s} b(\mathrm{a},\mathrm{Z})\mathrm{d}\mathrm{a}\mathrm{d}\mathrm{Z}\right)\mathrm{d}\mathrm{F}\mathrm{d}s+\int_{t_1}^{t}\int_{x_1}^{x} b(\mathrm{F},s)u(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s.$$

再利用 (6) 式，可以得到形如 (5) 式的估计

$$u(t,x)\leqslant (\tilde{a}(t,x)+\mathrm{U}\tilde{a}^m(t_1,x_1)\cdot \exp\left(\int_{t_0}^{t_1}\int_{x_0}^{x_1} b(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s\right)+\int_{t_0}^{t_1}\int_{x_0}^{x_1} b(\mathrm{F},s)\times \tilde{a}(\mathrm{F},s)\exp\left(\int_{t_0}^{\mathrm{F}}\int_{x_0}^{s} b(\mathrm{a},\mathrm{Z})\mathrm{d}\mathrm{a}\mathrm{d}\mathrm{Z}\right)\mathrm{d}\mathrm{F}\mathrm{d}s)\cdot \exp\left(\int_{t_1}^{t}\int_{x_1}^{x} b(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s\right)\leqslant \tilde{a}(t,x)(1+\mathrm{U}\tilde{a}^{m-1}(t_1,x_1))\exp\left(\int_{t_0}^{t}\int_{x_0}^{x} b(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s\right). \quad (7)$$

如果 $(t,x)\in \mathrm{K}_{i2}, i=3,4\cdots$，再利用 (7) 式及积分和不等式 (1) 得到

$$u(t,x)\leqslant \tilde{a}(t,x)+\mathrm{U}\tilde{a}^m(t_1,x_1)\cdot \exp\left(\int_{t_0}^{t_1}\int_{x_0}^{x_1} b(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s\right)+\int_{t_0}^{t_1}\int_{x_0}^{x_1} b(\mathrm{F},s)\tilde{a}(\mathrm{F},s)\cdot \exp\left(\int_{t_0}^{\mathrm{F}}\int_{x_0}^{s} b(\mathrm{a},\mathrm{Z})\mathrm{d}\mathrm{a}\mathrm{d}\mathrm{Z}\right)\mathrm{d}\mathrm{F}\mathrm{d}s+\int_{t_1}^{t_2}\int_{x_1}^{x} b(\mathrm{F},s)\tilde{a}(\mathrm{F},s)(1+\mathrm{U}\tilde{a}^{m-1}(t_1,x_1))\cdot \exp\left(\int_{t_0}^{\mathrm{F}}\int_{x_0}^{s} b(\mathrm{a},\mathrm{Z})\mathrm{d}\mathrm{a}\mathrm{d}\mathrm{Z}\right)\mathrm{d}\mathrm{F}\mathrm{d}s\leqslant \tilde{a}(t,x)(1+\mathrm{U}\tilde{a}^{m-1}(t_1,x_1))\exp\left(\int_{t_0}^{t_2}\int_{x_0}^{x} b(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s\right)=\tilde{a}(t,x)(1+\mathrm{U}\tilde{a}^{m-1}(t_1,x_1))\exp\left(\int_{t_0}^{t}\int_{x_0}^{x} b(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s\right).$$

如果 $(t,x)\in \mathrm{K}_{2j}, j=3,4\cdots$，又利用 (7) 式及积分和不等式 (1) 得到

$$u(t,x)\leqslant \tilde{a}(t,x)+\mathrm{U}\tilde{a}^m(t_1,x_1)\cdot \exp\left(\int_{t_0}^{t_1}\int_{x_0}^{x_1} b(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s\right)+\int_{t_0}^{t_1}\int_{x_0}^{x_1} b(\mathrm{F},s)\tilde{a}(\mathrm{F},s)\cdot \exp\left(\int_{t_0}^{\mathrm{F}}\int_{x_0}^{s} b(\mathrm{a},\mathrm{Z})\mathrm{d}\mathrm{a}\mathrm{d}\mathrm{Z}\right)\mathrm{d}\mathrm{F}\mathrm{d}s+\int_{t_1}^{t}\int_{x_1}^{x_2} b(\mathrm{F},s)\tilde{a}(\mathrm{F},s)(1+\mathrm{U}\tilde{a}^{m-1}(t_1,x_1))\exp\left(\int_{t_0}^{\mathrm{F}}\int_{x_0}^{s} b(\mathrm{a},\mathrm{Z})\mathrm{d}\mathrm{a}\mathrm{d}\mathrm{Z}\right)\mathrm{d}\mathrm{F}\mathrm{d}s\leqslant \tilde{a}(t,x)(1+\mathrm{U}\tilde{a}^{m-1}(t_1,x_1))\cdot \exp\left(\int_{t_0}^{t}\int_{x_0}^{x_2} b(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s\right)=\tilde{a}(t,x)(1+\mathrm{U}\tilde{a}^{m-1}(t_1,x_1))\exp\left(\int_{t_0}^{t}\int_{x_0}^{x} b(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s\right).$$

综上所述当 $(t,x)\in \mathrm{K}_{ik}\cup \mathrm{K}_{kj}, k=1,2, i,j=1,2,\cdots$ 时，估计式 (2) 成立.

假设 $(t,x)\in \mathrm{K}_{ik}\cup \mathrm{K}_{kj}, k>2, i,j=3,4\cdots$ 则积分和不等式 (1) 中未知函数有估计式

$$u(t,x)\leqslant \tilde{a}(t,x)\prod_{i=1}^{k-1}(1+\mathrm{U}\tilde{a}^{m-1}(t_i,x_i))\cdot \exp\left(\int_{t_0}^{t}\int_{x_0}^{x} b(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s\right). \quad (8)$$

当 $(t,x)\in \mathrm{K}_{i(k+1)}\cup \mathrm{K}_{(k+1)j}, k>2, i,j=3,4\cdots$ 时，由积分和不等式 (1) 推出

$$u(t,x)\leqslant \tilde{a}(t,x)+\int_{t_0}^{t_1}\int_{x_0}^{x_1} b(\mathrm{F},s)u(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s+\int_{t_1}^{t_2}\int_{x_1}^{x_2} b(\mathrm{F},s)u(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s+\cdots+\int_{t_{k-1}}^{t_k}\int_{x_{k-1}}^{x_k} b(\mathrm{F},s)u(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s+\int_{t_k}^{t}\int_{x_k}^{x} b(\mathrm{F},s)u(\mathrm{F},s)\mathrm{d}\mathrm{F}\mathrm{d}s+\sum_{i=1}^{k}\mathrm{U}_i u^m(t_i-0,x_i-0). \quad (9)$$

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$u_{i_k}\neq 0$,先关于 k 用数学归纳法证明不等式

$$\sum_{u_{i_1}\wedge\cdots\wedge u_{i_k}\neq 0} h_{i_1}\cdots h_{i_k} a_{i_1}\cdots a_{i_k} \geqslant \frac{n!}{(n-k)!\,n^k}\cdot \left(\sum_{i=1}^N a_i h_i\right)^k. \qquad (1)$$

若 $k=2$, $1\leqslant j\leqslant N$,由引理2.1,有

$$\sum_{i=1}^N a_i h_i = \sum_{u_j\wedge u_i\neq 0} h_i a_i + \sum_{u_j\wedge u_i=0} h_i a_i \leqslant \frac{n}{n-1}\cdot \sum_{u_j\wedge u_i\neq 0} h_i a_i,$$

$$\left(\sum_{i=1}^N a_i h_i\right)^2 = \sum_{i=1}^N\sum_{j=1}^N a_i a_j h_i h_j \leqslant \frac{n}{n-1}\cdot\sum_{u_j\wedge u_i\neq 0} a_i a_j h_i h_j,$$

故当 $k=2$ 时,(1)式成立.当 $2\leqslant k=m<n$ 时,(1)式成立,由引理2.1,有

$$\left(\sum_{i=1}^N a_i h_i\right)^{m+1} \leqslant \frac{(n-m)!\,n^m}{n!}\sum_{i=1}^N a_i h_i\cdot \sum_{u_{i_1}\wedge\cdots\wedge u_{i_m}\neq 0} h_{i_1}\cdots h_{i_m} a_{i_1}\cdots a_{i_m} = \frac{(n-m)!\,n^m}{n!}\cdot$$

$$\left[\sum_{u_j\wedge u_{i_1}\wedge\cdots\wedge u_{i_m}\neq 0} h_j h_{i_1}\cdots h_{i_m} a_j a_{i_1}\cdots a_{i_m} + \sum_{u_j\wedge u_{i_1}\wedge\cdots\wedge u_{i_m}=0,\ u_{i_1}\wedge\cdots\wedge u_{i_m}\neq 0} h_j h_{i_1}\cdots h_{i_m} a_j a_{i_1}\cdots a_{i_m}\right] \leqslant$$

$$\frac{(n-m)!\,n^m}{n!}\left(1+\frac{m}{n-m}\right)\sum_{u_j\wedge u_{i_1}\wedge\cdots\wedge u_{i_m}\neq 0} h_j h_{i_1}\cdots h_{i_m} a_j a_{i_1}\cdots a_{i_m}\leqslant \frac{(n-m-1)!\,n^{m+1}}{n!}\cdot$$

$$\sum_{u_j\wedge u_{i_1}\wedge\cdots\wedge u_{i_m}\neq 0} h_j h_{i_1}\cdots h_{i_m} a_j a_{i_1}\cdots a_{i_m},$$

即当 $k=m+1$ 时,(1)式成立.在(1)式中设 $k=n$ 即可以得出引理结论.又由引理2.1,可以知道等式成立当且仅当 P 为平行多面体.

定理2.1 若 P 为 R^n 中的关于原点对称的凸多面体,则 $U(P)\geqslant \frac{(n!)^{\frac{1}{n}}}{n}V(P)$,等式成立当且仅当 P 为平行多面体.

证明 由引理2.2,有

$$U(P) = \left(\frac{1}{n^n}\sum_{u_{i_1}\wedge\cdots\wedge u_{i_n}\neq 0} h_{i_1}\cdots h_{i_n} a_{i_1}\cdots a_{i_n}\right)^{\frac{1}{n}} \geqslant$$

$$\left[\frac{n!}{n^n\cdot n^n}\left(\sum_{i=1}^N a_i h_i\right)^n\right]^{\frac{1}{n}} = \frac{(n!)^{\frac{1}{n}}}{n}V(P),$$

等号成立当且仅当 P 为平行多面体.

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把(8)式代入(9)式中,考虑到 $(1+\mathrm{U}_j\tilde a^{m-1}(t_j,x_j))^m\leqslant(1+\mathrm{U}_j\tilde a^{m-1}(t_j,x_j))$ 则可以推出

$$u(t,x)\leqslant \tilde a(t,x)+\tilde a(t,x)\left[\exp\left(\int_{t_0}^{t_1}\int_{x_0}^{x_1} b(\mathrm{F},s)\,d\mathrm{F}\,ds\right)-1\right]+\tilde a(t,x)\left(1+\mathrm{U}_1 a^{m-1}(t_1,x_1)\right)\left[\exp\left(\int_{t_0}^{t_2}\int_{x_0}^{x_2} b(\mathrm{F},s)\,d\mathrm{F}\,ds\right)-\exp\left(\int_{t_0}^{t_1}\int_{x_0}^{x_1} b(\mathrm{F},s)\,d\mathrm{F}\,ds\right)\right]+\cdots+\tilde a(t,x)\prod_{i=1}^{k-1}\left(1+\mathrm{U}_i a^{m-1}(t_i,x_i)\right)\times$$

$$\left[\exp\left(\int_{t_0}^{t_k}\int_{x_0}^{x_k} b(\mathrm{F},s)\,d\mathrm{F}\,ds\right)-\exp\left(\int_{t_0}^{t_{k-1}}\int_{x_0}^{x_{k-1}} b(\mathrm{F},s)\,d\mathrm{F}\,ds\right)\right]+\sum_{i=1}^k \mathrm{U}_i\tilde a^m(t_i,x_i)\prod_{j=1}^{i-1}\left(1+\mathrm{U}_j a^{m-1}(t_j,x_j)\right)\cdot$$

$$\exp\left(\int_{t_0}^{t_i}\int_{x_0}^{x_i} b(\mathrm{F},s)\,d\mathrm{F}\,ds\right)+\int_{t_k}^{t}\int_{x_k}^{x} b(\mathrm{F},s)\cdot u(\mathrm{F},s)\,d\mathrm{F}\,ds=\tilde a(t,x)\prod_{i=1}^{k}\left(1+\mathrm{U}_i a^{m-1}(t_i,x_i)\right)\cdot$$

$$\exp\left(\int_{t_0}^{t_k}\int_{x_0}^{x_k} b(\mathrm{F},s)\,d\mathrm{F}\,ds\right)+\int_{t_k}^{t}\int_{x_k}^{x} b(\mathrm{F},s)u(\mathrm{F},s)\,d\mathrm{F}\,ds. \qquad (10)$$

类似于(3)式的处理方法,由(10)式可以得出结论.

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