Necessary and Sufficient Condition for Oscillation of Second Order Nonlinear Delay Differential Equations with Impulses*

一类二阶非线性脉冲时滞微分方程振动的充要条件

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Abstract: A necessary and sufficient condition for the oscillation of solutions of a class of second order nonlinear delay differential equations with impulses is obtained by using the contraction mapping principle and some differential inequalities.

Key words: differential equation, impulse, oscillation

摘要:利用压缩映射原理和微分不等式,得到一类二阶非线性脉冲时滞微分方程振动的1个充要条件.

关键词:微分方程 脉冲 振动

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Many systems in physics, chemistry, biology and information science have impulsive dynamical behaviors due to abrupt jumps at certain instants during the dynamical processes. These complex dynamical behaviors can be modeled by impulsive differential equations. The mathematical theory of impulsive differential equations has been developed by a large number of mathematicians[1]. Gopalsamy and Zhang[2] first investigated the properties of linear impulsive differential equations with a single delay. As we know, in spite of the large number of investigations of impulsive differential equations, the systematic theory of impulsive delay differential equations has not been established since the combined effects of time delay and discontinuity on the solutions are not easy to deal with. In recent years, some authors concentrated on the oscillation of this class of equations. Most of the obtained results are concerned with first order equations[3~10], while only a few are about second order equations[11~16]. Reference [14] studied the oscillation for linear equation:

$$\begin{cases} x''(t) + r(t)x'(t) + [p(t) - q(t)]x(t - \tau) = 0, t \geqslant 0, t \neq t_k, k = 1,2,\cdots, \\ x(t_k^+) = g_k(x(t_k)), x'(t_k^+) = h_k(x'(t_k)), \end{cases} \tag{0.1}$$

and obtained the sufficient conditions of oscillation by using differential inequalities. Reference [15] researched the following linear equation:

$$\begin{cases} x''(t) + \sum_{j=1}^{m} q_j x'(t - \sigma_j) + \sum_{i=1}^{n} p_i x'(t - \tau_i) = 0, t \geqslant 0, t \neq t_k, k = 1,2,\cdots, \\ x(t_k^+) - x(t_k) = b_k x(t_k), x'(t_k^+) - x'(t_k) = b_k x'(t_k), \end{cases} \tag{0.2}$$

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and got some necessary and sufficient conditions of non-oscillation of solutions based on the roots of the characteristic equation.

In this paper, we consider a class of second order nonlinear delay differential equations with impulses:

$$\begin{cases}[p(t)x'(t)]' + q(t)f(x(t-\tau)) = 0, \\ \quad t \neq t_k, k = 1,2,\cdots, \\ x(t_k^+) - x(t_k) = B_k x(t_k), x'(t_k^+) - \\ \quad x'(t_k) = C_k x'(t_k),\end{cases} \tag{0.3}$$

where $f(x) \in C^1(-\infty, +\infty)$, $p(t) \in C^1[0, +\infty)$, $q(t) \in C[0, +\infty)$; τ, B_k and C_k are constants, t_k is a given sequence. Our purpose is to find the necessary and sufficient conditions for oscillation of Equation (0.3).

1 Definitions and lemmas

Consider the impulsive delay differential equation:

$$\begin{cases}[p(t)x'(t)]' + q(t)f(x(t-\tau)) = 0, \\ \quad t \neq t_k, k = 1,2,\cdots, \\ x(t_k^+) - x(t_k) = B_k x(t_k), \\ \quad x'(t_k^+) - x'(t_k) = C_k x'(t_k),\end{cases} \tag{1.1}$$

where $\tau > 0, 0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim\limits_{t\to\infty} t_k = \infty$,

$$x'(t_k) = \lim_{h\to 0^-}\frac{x(t_k+h) - x(t_k)}{h}, x'(t_k^+) = \lim_{h\to 0^+}\frac{x(t_k+h) - x(t_k^+)}{h}. \tag{1.2}$$

Now we give the following preliminary notes, definitions and lemmas for further use.

$PC_\sigma = \{x: [\sigma-\tau,\sigma] \to R \mid x(t)$ is twice continuously differentiable for $t \in [\sigma-\tau,\sigma]\backslash\{t_k, k = 1,2,\cdots\}$; $x(t_k^+), x(t_k^-), x'(t_k^+), x'(t_k^-)$ exist and $x(t_k^-) = x(t_k), x'(t_k^-) = x'(t_k)$ for $t_k \in [\sigma-\tau,\sigma]\}$.

$\Omega_\sigma = \{x: [\sigma-\tau,\sigma] \to R \mid x(t)$ is continuous first for $t \neq t_k$; $x(t_k^+), x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$; $x(t)$ is continuously differentiable for $t \geqslant \sigma, t \neq t_k, t \neq t_k + \tau$; $x'(t_k^+), x'(t_k^-), x'(t_k^+ + \tau), x'(t_k^- + \tau)$ exist and $x'(t_k^-) = x'(t_k)\}$.

Definition 1.1 For any $\sigma \geqslant 0$ and $\phi \in PC_\sigma$, a function $x: [\sigma-\tau, +\infty) \to R$ is called a solution of Equation (1.1) satisfying the initial value condition

$$x(t) = \phi(t), \in [\sigma-\tau,\sigma], \tag{1.3}$$

if $x \in \Omega_\sigma$ and satisfies Equation (1.1) and (1.2).

In this paper, we only consider the nontrivial solutions of Equation (1.1) in $[t_0-\tau, +\infty)$.

Definition 1.2 A solution of Equation (1.1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called non-oscillatory.

Definition 1.3 Equation (1.1) is called oscillatory if all of its solutions are oscillatory. Otherwise, it is called non-oscillatory.

Lemma 1.1[1] Assume that (A_0) $m \in PC^1[R_+, R]$ and $m(t)$ is left-continuous at $t_k, k = 1,2,\cdots$; (A_1) For $k = 1,2,\cdots, t \geqslant t_0, m'(t) \leqslant p(t)m(t) + q(t), t \neq t_k, m(t_k^+) \leqslant d_k m(t_k) + b_k$, where $q, p \in PC^1[R_+, R]$, $d_k \geqslant 0$ and b_k are constants. $PC[R_+, R]$ denote the class of piecewise continuous functions from R_+ to R, with discontinuities of the first kind only at $t = t_k, k = 1,2,\cdots$. Then

$$m(t) \leqslant m(t_0)\prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^t p(s)\mathrm{d}s\right) + \sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^t p(s)\mathrm{d}s\right)\right)b_k + \int_{t_0}^t \sum_{s<t_k<t} d_k \exp\left(\int_s^t p(\sigma)\mathrm{d}\sigma\right)q(s)\mathrm{d}s, t \geqslant t_0.$$

2 The main results

Consider the equation

$$\begin{cases}[p(t)x'(t)]' + q(t)f(x(t-\tau)) = 0, \\ \quad t \neq t_k, k = 1,2,\cdots, \\ x(t_k^+) - x(t_k) = B_k x(t_k), \\ x'(t_k^+) - x'(t_k) = C_k x'(t_k), \\ x(t_0^+) = x_0, x'(t_0^+) = x'_0.\end{cases} \tag{2.1}$$

We assume that the following conditions ($H_1 \sim H_3$) hold.

(H_1) $p'(t) \geqslant 0, p(t) > 0, q(t) \geqslant 0$,

(H_2) $xf(x) > 0, x \neq 0, f'(x) \geqslant 0, f(0) = 0, 0 \leqslant c \leqslant \dfrac{f(x)}{x} \leqslant d$,

(H_3) $B_k \in (-1, 0), C_k \in (-1, +\infty)$.

Remark 2.1 In view of $x(t_k^+) = (1 + B_k)\cdot x(t_k)$, Equation (2.1) is oscillatory when $B_k < -1$.

Lemma 2.1 Assume that conditions ($H_1 \sim H_3$) hold. Furthermore, suppose that $x(t)$ is a solution of Equation (2.1). Then there exists $T \geqslant t_0$ such that $x(t) > 0$ for $t \geqslant T$.

Proof We first prove that $x'(t_k) \geqslant 0$ for any $t_k \geqslant T$. If it is not true, then there exists some j such that $x'(t_j) < 0$ when $t_j \geqslant T + \tau$. Thus

$$x'(^+_j) = (1 + B_j)x'(t_j) < 0. \tag{2.2}$$

Set $S(t) = p(t)x'(t)$ and $p(t_j^+)x'(t_j^+) = p(t_j)x'(t_j^+) = -A, A > 0$. If $t \in (t_{j+i-1}, t_{j+i}]$, then $t - \tau > T$ and $x(t-\tau) > 0$. It follows from condition (H_2) that $f(x(t-\tau)) > 0$ and $S'(t) = -q(t)f(x(t-\tau)) \leqslant 0$. $S(t)$ decreases in the interval $(t_{j+i-1}, t_{j+i}]$, then $p(t_{j+1})x'(t_{j+1}) \leqslant p(t_j^+)x'(t_j^+) = -A < 0$, $p(t_{j+2})x'(t_{j+2}) \leqslant p(t_{j+1}^+)x'(t_{j+1}^+) = p(t_{j+1})(1 + C_{j+1})x'(t_{j+1}) \leqslant -(1+C_{j+1})A < 0$.

Using the method of deduction, we obtain

$$S(t) = p(t)x'(t) \leqslant -\prod_{t_j<t_k<t}(1+C_k)A < 0,$$

$$t \in (t_{j+n}, t_{j+n+1}], \tag{2.3}$$

and then $x'(t) \leqslant -A\dfrac{\prod_{t_j<t_k<t}(1+C_k)}{p(t)}$. Using Lemma 1.1 and in view of $x(t_k^+) = (1+B_k)x(t_k)$, we have $x(t) \leqslant x'(t_j^+)\prod_{t_j<t_k<t}(1+B_k) - A\int_{t_j}^{t}\prod_{s<t_k<t}(1+B_k)\dfrac{\prod_{t_j<t_k<s}(1+C_k)}{p(s)}\mathrm{d}s$,

$$x(t) \leqslant \prod_{t_j<t_k<t}(1+B_k)(x'(t_j^+) - A\int_{t_j}^{t}\prod_{t_{j}t_j<t_k<s}\frac{1+C_k}{1+B_k}\frac{1}{p(s)}\mathrm{d}s), t > t_j. \tag{2.4}$$

It follows from Inequation (2.3) that there exists $T_1 > 0$ such that $x(t) \leqslant 0$ for $t > T_1$ in view of Inequation (2.2) and $\int_{t_j}^{t}\prod_{t_{j}t_j<t_k<s}\dfrac{1+C_k}{1+B_k}\dfrac{1}{p(s)}\mathrm{d}s > 0$. This is a contradiction.

So $x'(t_k) \geqslant 0$ for any $t_k \geqslant T$. Then $x'(t_k^+) = (1 + C_k)x'(t_k)$ and $S(t_{k+1}) = p(t_{k+1})x'(t_{k+1}) \geqslant 0$. Counting the fact that $S(t)$ decreases in the interval $(t_{j+i-1}, t_{j+i}]$, we have $S(t) = p(t)x'(t) \geqslant 0$ as $t \in (t_k, t_{k+1}]$, $t_k \geqslant T$, namely $x'(t) \geqslant 0$. Lemma 2.1 is then proved.

Remark 2.2 If the solution $x(t)$ in Theorem 2.1 is eventually negative, then $x'(t_k^+) \leqslant 0$ and $x'(t) \leqslant 0$ for $t \in (t_k, t_{k+1}]$, where $t_k \geqslant T$.

Theorem 2.1 Assume that conditions ($H_1 \sim H_3$) hold. Further suppose that

$$\int_{t_0}^{+\infty}\prod_{t_0<t_k<s}(1+C_k)q(s)\mathrm{d}s = +\infty. \tag{2.5}$$

Then every solution of Equation (2.1) is oscillatory.

Proof If Theorem 2.1 is not true, then Equation (2.1) has a non-oscillatory solution $x(t)$. We may assume that $x(t) > 0$ for $t \geqslant T$ without loss of generality. It follows from Lemma 2.1 that $x(t_k^+) \geqslant 0$ and $x(t) \geqslant 0$ for $t \in (t_k, t_{k+1}]$, where $t_k \geqslant T$. Let

$$w(t) = \frac{p(t)x'(t)}{f(x(t-\tau))}. \tag{2.6}$$

Then $w(t) \geqslant 0$ for $t \geqslant T - \tau$. If there exists some $t_{j(k)}$ such that $t_k - \tau = t_{j(k)}$, then $x(t_k^+ - \tau) = (1 + B_{j(k)})x(t-\tau)$. Otherwise, $x(t_k^+ - \tau) = x(t-\tau)$. So $x(t_k^+ - \tau) \geqslant (1 + B_{j(k)})x(t-\tau)$. Then $f(x(t_k^+ - \tau)) \geqslant f((1+B_{j(k)})x(t-\tau))$ in view of $f'(x) > 0$. Due to $0 \leqslant c \leqslant \dfrac{f(x)}{x^n} \leqslant d$ and $f((1+B_{j(k)})x(t_k)) \geqslant \dfrac{c(1+B_{j(k)})^n}{d}f(x(t_k))$, we obtain $w(t_k^+) = \dfrac{p(t_k^+)x'(t_k^+)}{f(x(t_k^+ - \tau))} \leqslant \dfrac{p(t_k)(1+C_k)x'(t_k)}{f((1+B_{j(k)})x(t_k-\tau))} \leqslant \dfrac{d(1+C_k)}{c(1+B_{j(k)})^n}\dfrac{p(t_k)x'(t_k)}{f(x(t_k-\tau))}$,

$$w'(t) = [-q(t)f^2(x(t-\tau) - p(t)x'(t)f'(x(t-\tau))x'(t-\tau)]/[f^2(x(t-\tau))] \leqslant -q(t), t \neq t_k, t \geqslant T + \tau. \tag{2.7}$$

It follows from Lemma 1.1 that

$$w(t) \leqslant -\int_{t_0}^{t}\prod_{t_{0s}<t_k<t}\frac{d(1+C_k)}{c(1+B_{j(k)})^n}q(s)\mathrm{d}s. \tag{2.8}$$

It is seen that $\dfrac{d(1+C_k)}{c(1+B_{j(k)})^n}q(s) \geqslant (1+C_k)q(s)$ since $B_{j(k)} \in (-1, 0)$, $n > 2$ and $C_k \in (-1, +\infty)$. If Equation (2.5) holds, let $t \to +\infty$ in Inequation (2.8), then $\lim_{t\to+\infty} w(t) = -\infty$. This is a contradiction. Then the proof is complete.

Now we seek the necessary and sufficient conditions of oscillation for Equation (2.1). The conditions ($H_1 \sim H_3$) are replaced by the conditions ($H_1^* \sim H_3^*$) respectively as follows:

(H_1^*) $p'(t) \geqslant 0, \dfrac{1}{2} < a \leqslant p(t) \leqslant b < 1$, $q(t) \geqslant 0$,

(H_2^*) $xf(x) > 0, x \neq 0, f'(x) \geqslant 0, f(0) = 0, 0 \leqslant c \leqslant \dfrac{f(x)}{x} \leqslant d$,

(H_3^*) $B_k = C_k \in (-1, 0)$.

Lemma 2.2 Assume that conditions ($H_1^* \sim H_3^*$) hold and that for some $m \geqslant 1$,

$$\frac{c}{d}\prod_{t-m\tau\leqslant t_k<t}(1+B_k)^{-1} \leqslant 1, t - m\tau > t_0. \tag{2.9}$$

Then Equation (2.1) is oscillatory if and only if the equation without impulses

$$[p(t)y'(t)]' + Q(t)f(y(t-\tau)) = 0 \tag{2.1*}$$

is oscillatory, where $Q(t) = \dfrac{c}{d}\prod_{t-\tau\leqslant t_k<t}(1+B_k)^{-1}q(t)$.

Proof To prove Lemma 2.2, we introduce the following delay differential inequalities with and without impulses(denoting by star) respectively.

$$\begin{cases} [p(t)x'(t)]' + q(t)f(x(t-\tau)) \leqslant 0, t \neq t_k, \\ \quad k = 1,2,\cdots, \\ x(t_k^+) - x(t_k) = B_k x(t_k), x'(t_k^+) - x'(t_k) = \\ \quad C_k x'(t_k). \end{cases} \tag{2.10}$$

$$[p(t)y'(t)]' + Q(t)f(y(t-\tau)) \leqslant 0; \tag{2.10*}$$

$$\begin{cases} [p(t)x'(t)]' + q(t)f(x(t-\tau)) \geqslant 0, t \neq t_k, \\ \quad k = 1,2,\cdots, \\ x(t_k^+) - x(t_k) = B_k x(t_k), x'(t_k^+) - x'(t_k) = \\ \quad C_k x'(t_k). \end{cases} \tag{2.11}$$

$$[p(t)y'(t)]' + Q(t)f(y(t-\tau)) \geqslant 0. \tag{2.11*}$$

We claim that: (i) Inequation (2.10) has no eventually positive solution if and only if the corresponding Inequation (2.10*) has no eventually positive solution. (ii) Inequation (2.11) has no eventually positive solution if and only if the corresponding Inequation (2.11*) has no eventually positive solution.

We first prove (i). Let $x(t)$ be an eventually positive solution of Inequation (2.10). Then there exists $T \geqslant 0$ such that $x(t) > 0$ and $x(t-\tau) > 0$ for $t \geqslant T$. From condition (H_2), $f(x(t-\tau)) > 0$ and $q(t)f(x(t-\tau)) \geqslant 0$ for $t \geqslant T$.

Set $y(t) = \frac{c}{d}\prod_{t-m\tau\leqslant t_k<t}(1+B_k)^{-1}x(t)$. Hence $y(t) > 0$ and $y(t-\tau) > 0$ for $t \geqslant T$. It follows from Inequation (2.9) that $y(t) \leqslant x(t)$. Then $f(y(t-\tau)) \leqslant f(x(t-\tau))$ in view of $f'(x) \geqslant 0$. From condition (H_3^*), $B_k \in (-1,0)$ implies that $\prod_{t-m\tau\leqslant t_k<t}(1+B_k)^{-1} \geqslant \prod_{t-\tau\leqslant t_k<t}(1+B_k)^{-1}$. Then for $t \neq t_k$, we have

$$[p(t)y'(t)]' + Q(t)f(y(t-\tau)) = \frac{c}{d}\prod_{t-m\tau\leqslant t_k<t}(1+B_k)^{-1}[p(t)x'(t)]' + \frac{c}{d}\prod_{t-\tau\leqslant t_k<t}(1+B_k)^{-1}q(t)f(y(t-\tau)) \leqslant \frac{c}{d}\prod_{t-m\tau\leqslant t_k<t}(1+B_k)^{-1}[p(t)x'(t)]' + \frac{c}{d}\prod_{t-m\tau\leqslant t_k<t}(1+B_k)^{-1}q(t)f(y(t-\tau)) = \frac{c}{d}\prod_{t-m\tau\leqslant t_k<t}(1+B_k)^{-1}\cdot([p(t)x'(t)]' + q(t)f(x(t-\tau))).$$

Taking account of that $x(t)$ is a solution of Inequation (2.10), then

$$[p(t)y'(t)]' + Q(t)f(y(t-\tau)) \leqslant 0, t \neq t_k. \tag{2.12}$$

For $t_k \geqslant T$, we have

$$y(t_k^+) = \frac{c}{d}\prod_{t-m\tau\leqslant t_j<t_k^+}(1+B_j)^{-1}x(t_k^+) = \frac{c}{d}\prod_{t-m\tau\leqslant t_j\leqslant t_k}(1+B_j)^{-1}(1+B_k)x(t_k) = \frac{c}{d}\prod_{t-m\tau\leqslant t_j<t_k}(1+B_j)^{-1}x(t_k) = y(t_k),$$

$$y(t_k^-) = \frac{c}{d}\prod_{t-m\tau\leqslant t_j<t_k^-}(1+B_j)^{-1}x(t_k^-) = \frac{c}{d}\prod_{t-m\tau\leqslant t_j<t_k}(1+B_j)^{-1}x(t_k) = y(t_k).$$

It follows from the above that $y(t)$ is continuous on $[T, +\infty)$ and is an eventually positive solution of Inequation (2.10*). Conversely, let $y(t)$ be an eventually positive solution of Inequation (2.10*) and $y(t) > 0, y(t-\tau) > 0$ for $t \geqslant T$, then $[p(t)y'(t)]' = -Q(t)f(y(t-\tau)) \leqslant 0$. From condition$(H_2)$, $0 \leqslant c \leqslant \frac{f(x)}{x^n} \leqslant d$, it implies that

$$[p(t)y'(t)]' + \frac{c^2}{d}\prod_{t-\tau\leqslant t_k<t}(1+B_k)^{-1}q(t)y^n(t-\tau) \leqslant 0. \tag{2.13}$$

Set $x(t) = \frac{c}{d}\prod_{T\leqslant t_k<t}(1+B_k)y(t)$. For $t \neq t_k$, it follows from the fact $0 < \frac{c}{d} \leqslant 1, n \geqslant 3$ and $0 < 1 + B_k < 1$ that

$$[p(t)x'(t)]' + q(t)f(x(t-\tau)) \leqslant \frac{c}{d}\prod_{T\leqslant t_k<t}(1+B_k)[p(t)y'(t)]' + q(t)\cdot d\cdot x^n(t-\tau) = \frac{c}{d}\prod_{T\leqslant t_k<t}(1+B_k)[p(t)y'(t)]' + q(t)\cdot d\cdot(\frac{c}{d})^n\prod_{T\leqslant t_k<t-\tau}(1+B_k)^n y^n(t-\tau) \leqslant \frac{c}{d}\prod_{T\leqslant t_k<t}(1+B_k)[p(t)y'(t)]' + q(t)(\frac{c}{d})^3\cdot d\cdot\prod_{T\leqslant t_k<t-\tau}(1+B_k)y^n(t-\tau) = \frac{c}{d}\prod_{T\leqslant t_k<t}(1+B_k)([p(t)y'(t)]' + \frac{c^2}{d}\prod_{t-\tau\leqslant t_k<t}(1+B_k)^{-1}q(t)y^n(t-\tau)).$$

From Inequation (2.13) we have

$$[p(t)x'(t)]' + q(t)f(x(t-\tau)) \leqslant 0, t \neq t_k. \tag{2.14}$$

On the other hand, for any $t_k \geqslant T$,

$$x(t_k^+) = \frac{c}{d}\prod_{T\leqslant t_j<t_k^+}(1+B_j)y(t_k^+) = \frac{c}{d}\prod_{T\leqslant t_j\leqslant t_k}(1+B_j)y(t_k) = \frac{c}{d}\prod_{T\leqslant t_j<t_k}(1+B_j)y(t_k)(1+B_k) = (1+B_k)x(t_k).$$

Similarly, it is easy to calculate that $x'(t_k^+) = (1+B_k)x'(t_k)$ by using Equation (1.2). Hence we can see that $x(t) = \frac{c}{d}\prod_{T\leqslant t_k<t}(1+B_k)y(t)$ is an eventually positive solution of Equation (2.10). (i) is then proved. The proof of (ii) is similar to that of (i). The proof of Lemma 2.2 is then completed.

Lemma 2.3 Assume that conditions ($H_1^* \sim H_3^*$) hold, then Equation (2.1*) is oscillatory if and only if

$$\int_\epsilon^\infty tQ(t)\mathrm{d}t = +\infty, \tag{2.15}$$

where ϵ is a small positive number.

Proof To prove the sufficiency, without loss of generality we may assume that $y(t)$ is a eventually positive solution of Equation (2.1*), then there exists a $T>0$ such that $y(t)>0$ and $y(t-\tau)>0$ for $t>T$. Since $Q(t)\geqslant 0$ and $f(y(t-\tau))>0$, Equation (2.1*) implies that $p(t)y'(t)\geqslant 0$ and $p(t)y'(t)$ is non-increasing. Let $\lim_{t\to+\infty} y'(t) = l$. Suppose that $l<0$, then $\lim_{t\to+\infty} y(t) = -\infty$. This is contradict to $y(t)$ is a eventually positive solution of Equation (2.1*). So $y'(t)\geqslant 0, y'(t-\tau)\geqslant 0$ for $t>T$.

Set $w(t) = \frac{tp(t)y'(t)}{f(y(t-\tau))}, (t\geqslant T)$. Notice that $p(t)>0$ and $f(y(t-\tau)>0), w(t)>0$. From Equation (2.1*), we have $[p(t)x'(t)]'\leqslant 0$ and $p(t)y'(t)\leqslant p(t-\tau)y'(t-\tau)$. Therefore,

$$w'(t) = \frac{p(t)y'(t)}{f(y(t-\tau))} + t\cdot \frac{-Q(t)f^2(y(t-\tau)-p(t)y'(t)f'(y(t-\tau))y'(t-\tau)}{f^2(y(t-\tau))}$$

and

$$w'(t) + tQ(t) = \frac{p(t)y'(t)}{f(y(t-\tau))} - \frac{tp(t)y'(t)f'(y(t-\tau))y'(t-\tau)}{f^2(y(t-\tau))} \leqslant \frac{p(t)y'(t)}{f(y(t-\tau))} \leqslant \frac{p(t-\tau)y'(t-\tau)}{f(y(t-\tau))} \leqslant \frac{by'(t-\tau)}{f(y(t-\tau))},$$

namely,

$$w'(t) + tQ(t) \leqslant \frac{by'(t-\tau)}{f(y(t-\tau))}. \tag{2.16}$$

Since $0\leqslant c\leqslant \frac{f(x)}{x^n}\leqslant d, \frac{1}{dy^n}\leqslant\frac{1}{f(y)}\leqslant\frac{1}{cy^n}$ and

$$0<\int_T^{+\infty}\frac{1}{f(y)}\mathrm{d}s<+\infty. \tag{2.17}$$

Now integrate Inequation (2.16) from T to t,

$$w(t) + \int_T^t sQ(s)\mathrm{d}s \leqslant w(T) + \int_T^t \frac{by'(s-\tau)}{f(y(s-\tau))}\mathrm{d}s = w(T) + b\int_T^t \frac{1}{f(y(s-\tau))}\mathrm{d}(y(s-\tau)) = w(T) + b\int_{y(T-\tau)}^{+\infty}\frac{\mathrm{d}u}{f(u)} - b\int_{y(t-\tau)}^{+\infty}\frac{\mathrm{d}u}{f(u)}.$$

In view of $w(t)>0$ and Inequation (2.17), we obtain

$$\int_T^t sQ(s)\mathrm{d}s \leqslant w(T) + b\int_{y(T-\tau)}^{+\infty}\frac{\mathrm{d}u}{f(u)} := M_0. \tag{2.18}$$

Letting $t\to 0$ in Inq. (2.18), we have $\int_T^{+\infty} sQ(s)\mathrm{d}s \leqslant M_0$. It follows from $q(t)\in C[0,+\infty)$ and $q(t)\geqslant 0$ that $\int_\epsilon^T sQ(s)\mathrm{d}s<+\infty$. So $\int_\epsilon^{+\infty} sQ(s)\mathrm{d}s<+\infty$. This contradicts to Equation (2.15). Then we proves the sufficiency part in Lemma 2.3.

Next, we use the contraction mapping principle to prove the necessity by contradiction. Now assume that Equation (2.15) fails. Let Ω be the space of solutions of Equation (2.1*) and set $Y = \{y \mid y\in\Omega, \frac{1}{2b}\leqslant y(t) \leqslant \frac{1}{2a-b}\}$. Define an operator $F: Y\to Y$ by

$$Fy(t) = \frac{1}{p(t)} + \frac{1}{p(t)}\left[\int_{t_0}^t y(s)p'(s)\mathrm{d}s - \int_t^{+\infty}(s-t)Q(s)f(y(s-\tau))\mathrm{d}s\right]. \tag{2.19}$$

Set $L = \sup_{\frac{1}{2b}\leqslant y(t)\leqslant\frac{1}{2a-b}} f'(y)$. Choose t_0 large enough so that $L\int_{t_0}^{+\infty} sQ(s)\mathrm{d}s \leqslant \frac{a(2a-b)}{2b}$ by taking account of Equation (2.15) fails, then from Equation (2.19) we have

$$Fy(t) \geqslant \frac{1}{p(t)} - \frac{1}{p(t)}\int_t^{+\infty}(s-t)Q(s)f(y(s-\tau))\mathrm{d}s \geqslant \frac{1}{b} - \frac{L}{a(2a-b)}\int_{t_0}^{+\infty}sQ(s)\mathrm{d}s \geqslant \frac{1}{b} - \frac{1}{a(2a-b)}\times\frac{a(2a-b)}{2b} = \frac{1}{2b},$$

and

$$Fy(t) \leqslant \frac{1}{p(t)} + \frac{1}{p(t)}\int_{t_0}^t y(s)p'(s)\mathrm{d}s \leqslant \frac{1}{p(t)} + \frac{p(t)-p(t_0)}{p(t)}\cdot\frac{1}{2a-b} \leqslant \frac{1}{a} + \frac{b-a}{a}\cdot\frac{1}{2a-b} = \frac{1}{2a-b}.$$

From the above result, $F(Y)\subseteq Y$. On the other hand, for any $y_1(t), y_2(t)\in Y$,

$$|Fy_1(t) - Fy_2(t)| \leqslant \frac{1}{p(t)}\|y_1 - y_2\|\int_{t_0}^t p'(s)\mathrm{d}s +$$

$L\|y_1 - y_2\|\int_t^{+\infty} sQ(s)\mathrm{d}s = \frac{p(t) - p(t_0)}{p(t)}\|y_1 - y_2\| + L\|y_1 - y_2\|\int_t^{+\infty} sQ(s)\mathrm{d}s \leqslant \frac{b-a}{a}\|y_1 - y_2\| + \frac{a(2a-b)}{2b}\|y_1 - y_2\| < (\frac{b-a}{a} + \frac{2a-b}{2b})\|y_1 - y_2\| \leqslant \frac{b}{2a}\|y_1 - y_2\|$.

From condition (H_1^*), we have $0 < \frac{b}{2a} < 1$, and then F is a contraction map. It follows from the contraction mapping principle that F has a fixed point in Y since $F(Y) \subseteq Y$. So there exists $y(t) \in Y$ such that

$$y(t) = \frac{1}{p(t)} + \frac{1}{p(t)}[\int_{t_0}^t y(s)p'(s)\mathrm{d}s - \int_t^{+\infty}(s - t)Q(s)f(y(s-\tau))\mathrm{d}s]. \tag{2.20}$$

From Equation (2.20), we have $p(t)y(t) = 1 + \int_{t_0}^t y(s)p'(s)\mathrm{d}s - \int_t^{+\infty}(s - t)Q(s)f(y(s-\tau))\mathrm{d}s$, $p'(t)y(t) + p(t)y'(t) = y(t)p'(t) - \int_t^{+\infty} - Q(s)f(y(s-\tau))\mathrm{d}s$, $p(t)y'(t) = \int_t^{+\infty} Q(s)f(y(s-\tau))\mathrm{d}s$, $[p(t)y'(t)]' = -Q(t)f(y(t-\tau))$, $[p(t)y'(t)]' + Q(t)f(y(t-\tau)) = 0$.

Hence Equation (2.1^*) has a solution $y(t) \in Y = \{y | y \in \Omega, \frac{1}{2b} \leqslant y(t) \leqslant \frac{1}{2a-b}\}$. This is a contradiction to that Equation (2.1^*) is oscillatory. So Equation (2.15) holds and the necessity is proved. The proof of Lemma 2.3 is completed.

We obtain from lemma 2.2 and lemma 2.3 the following theorem.

Theorem 2.2 Assume that conditions ($H_1^* \sim H_3^*$) and (2.9) hold. Then Equation (2.1) is oscillatory if and only if

$$\int_\varepsilon^{+\infty} \frac{c}{d}\prod_{t-\tau\leqslant t_k<t}(1 + B_k)^{-1}tq(t)\mathrm{d}t = +\infty. \tag{2.21}$$

3 Example

To illustrate our results, we consider the example:

$$\begin{cases} [(\frac{3}{5} + \frac{3}{5\pi}\arctan(t))x'(t)]' + \\ \quad t^2 f(x(t-1)) = 0, t \neq k, k = 1,2,\cdots, \\ x(k^+) - x(k) = -\frac{1}{2}x(k), x'(k^+) - x'(k) = \\ \quad -\frac{1}{2}x'(k), x(0^+) = x_0, x'(0^+) = x'_0, \end{cases} \tag{3.1}$$

where

$$f(x) = \begin{cases} x^3\exp(\sin(\ln|x|)), x \neq 0, \\ 0, x = 0. \end{cases}$$

In Equation (3.1), $p(t) = (\frac{3}{5} + \frac{3}{5\pi}\arctan(t))$, $q(t) = t^2, B_k = C_k = -\frac{1}{2}\tau = 1, t_k = k$. We can calculate that $p'(t) \geqslant 0$ and $\frac{3}{5} \leqslant p(t) \leqslant \frac{9}{10}$ for $t \geqslant 0$, $xf(x) > 0$ and $f'(x) = x^2\exp(\sin(\ln|x|))(3 + \cos(\ln|x|)) > 0$ for $x \neq 0$, $f'(0) = 0$, $\exp(-1) \leqslant \frac{f(x)}{x^3} \leqslant \exp(1)$. We can also prove that $f(x) \in C^1(-\infty, +\infty)$. So conditions ($H_1^* \sim H_3^*$) hold. Hence conditions (2.9) and (2.21) are valid in view of

$$\frac{c}{d}\prod_{t-\tau\leqslant t_k<t}(1 + B_k)^{-1} = \frac{\exp(-1)}{\exp(1)}\prod_{t-1\leqslant k<t}(1 - \frac{1}{2})^{-1} \leqslant \frac{2}{\exp(2)} < 1,$$

$$\int_\varepsilon^\infty \frac{c}{d}\prod_{t-\tau\leqslant t_k<t}(1 + B_k)^{-1}tq(t)\mathrm{d}t = \int_\varepsilon^\infty \frac{c}{d}\prod_{t-1\leqslant k<t}(1 - \frac{1}{2})^{-1}tq(t)\mathrm{d}t \geqslant \int_\varepsilon^\infty \frac{t^3}{\exp(2)}\mathrm{d}t = +\infty.$$

From Theorem 2.2, we conclude that Equation (3.1) is oscillatory.

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(上接第245页 Continue from page 245)

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