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Contractible Edges of k-connected Graphs *

k-连通图的可收缩边

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Abstract: This paper show that a k-connected graph G has at least two contractible edges if any fragment which neighborhood contains an edge has cardinality exceeding $\frac{k}{4}$.

Key words: k-connected graph, contractible edge, fragment

摘要:证明了对 k-连通图 G,若 G 的任意一个断片满足当 $N(F)$ 中含有边就有 $|F|>\frac{k}{4}$,则 G 至少有 2 条可收缩边.

关键词:k-连通图　可收缩边　断片

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Let k be a positive integer, G a k-connected graph. An edge of G is said to be a k-contractible edge if its contraction yields again a k-connected graph. By Tutte's famous result[1], any 3-connected graph with order at least 5 has a 3-contractible edge. But for $k \geqslant 4$, Thomassen[2] show that there are infinitely many k-connected k-regular graphs which do not have any k-contractible edge. So, the contraction-critical k-connected graph for $k \geqslant 4$ was introduced, which is the non-complete k-connected graph without k-contractible edges. The contraction-critical 4-connected graphs are characterized, which are two special classes of 4-regular graphs. For $k \geqslant 5$, the characterization of contraction-critical k-connected graphs seems to be very hard. In general, Y. Egawa[3] show that every k-connected graph with minimum degree more than or equal to $\lfloor \frac{5k}{4} \rfloor$ has a contractible edge. Later, M. Kriesell[4] show that every $k(k \geqslant 4)$-connected graph such that any fragment whose neighborhood contains an edge has cardinality exceeding $\frac{k}{4}$ has a contractible edge. This improve the result of Y. Egawa. In this paper, we will show that every $k(k \geqslant 4)$-connected graph such that any fragment whose neighborhood contains an edge has cardinality exceeding $\frac{k}{4}$ has at least two contractible edges.

1 Preliminaries

We only consider finite simple undirected graph. For terms not defined here we refer the reader to the reference[5]. Let $G=(V(G),E(G))$ be a graph, $V(G)$ denotes the vertex set and $E(G)$ the edge set. Let $|G|=|V(G)|$, $\kappa(G)$ denote the vertex connectivity of G. An edge joining the vertex x, y will be written as xy. By $E_k(G)$, we denote the collection of all k-contractible edges in k-connected graph G. For $x \in F \subseteq V(G)$, we define $N_G(x)=\{y \mid xy \in E(G)\}$. By $d_G(x)=|N_G(x)|$, we denote the degree of x. Let $N_G(F)=\bigcup_{x\in F} N_G(x)-F$. A set $T \subseteq V(G)$ is called a separating set of a connected

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graph G, if $G-T$ has at least two connected components. A separating set with $\kappa(G)$ vertices is called a smallest separating set. Let G be a non-complete graph, T a smallest separating set. The union of at least one but not of all the components of $G-T$ is called a T-fragment. A fragment of G is a T-fragment for some smallest separating set T. Let $F \subseteq V(G)$ be a T-fragment. Then, $\overline{F}=V(G)-(F \cup T) \neq \varnothing$, and $\overline{F}$ is also a T-fragment and $N_G(F)=T=N_G(\overline{F})$. The set of all smallest separating sets of G will be denoted by $\mathcal{T}_G$. We often omit the index G if it is clear from the context.

We need more definitions introduced in reference [6]. For a graph G, let $\mathcal{S}$ be a non-empty set of subset of $V(G)$. An $\mathcal{S}$-fragment of G is a T-fragment of G for any $T \in \mathcal{T}_G$ such that there is an $S \in \mathcal{S}$ with $S \subseteq T$. An inclusion-minimal $\mathcal{S}$-fragment of G is called an $\mathcal{S}$-end and one of the least vertex numbers is an $\mathcal{S}$-atom. A graph G is called $\mathcal{S}$-critical if for each $S \in \mathcal{S}$ there is $T \in \mathcal{T}_G$ such that $S \subseteq T$, and for any $\mathcal{S}$-fragment F there is a $T' \in \mathcal{T}_G$ such that $T' \cap F \neq \varnothing$, and $T' \cap (F \cup N(F))$ contains an element of $\mathcal{S}$.

The following properties of fragments are folklore[6], we will use them without any further reference.

Let T, $T' \in \mathcal{T}_G$, and F, F' be the T, T'-fragment of G, respectively. If $F \cap F' \neq \varnothing$, then

$$|F \cap T'| \geqslant |\overline{F'} \cap T|, \quad |F' \cap T| \geqslant |\overline{F} \cap T'|. \tag{1}$$

If $F \cap F' \neq \varnothing \neq \overline{F} \cap \overline{F'}$, then both $F \cap F'$ and $\overline{F} \cap \overline{F'}$ are fragments of G, and $N(F \cap F')=(F' \cap T) \cup (T' \cap T) \cup (F \cap T')$. If $F \cap F' \neq \varnothing$ and $F \cap F'$ is not a fragment of G, then $\overline{F} \cap \overline{F'}=\varnothing$ and

$$|F \cap T'| > |\overline{F'} \cap T|, \quad |F' \cap T| > |\overline{F} \cap T'|. \tag{2}$$

Also, by definition, the two end vertices of any edge which is not k-contractible is contained in some smallest separating sets. For an edge e of G, a fragment A of G is said to be a fragment with respect to e if $V(e) \subseteq N(A)$.

Theorem 1[4] Let G be a $k(k \geqslant 4)$-connected graph such that any fragment whose neighborhood contains an edge has cardinality exceeding $\frac{k}{4}$, then, G has contractible edge.

Then, for $4 \leqslant k \leqslant 7$, every contraction-critical k-connected graph contains a vertex of degree k. It is very interesting to study the properties of contraction critical k-connected graphs and the distribution of k-contractible edges. N. Dean show following

Theorem 2[7] For every k-connected graph such that any fragment whose neighborhood contains an edge has cardinality exceeding $\frac{k}{2}$, the collection of k-contractible edges of G induced a 2-connected spanning subgraph of G.

So the following problem was given.

Problem 1 Let G be a k-connected graph such that any fragments whose neighborhood contains an edge has cardinality exceeding $\frac{k}{4}$. If the subgraph $H=(V(G), E_k(G))$ is formed by $V(G)$ and the k-contractible edges of G is connected.

In reference [8], we show that it is true for $k=4$, but for $k \geqslant 5$ it is false.

Further, in reference [8] we give a lower bound for the number of contractible edges of such a 5-connected graph. This paper, for general k, we determine the lower bound of the number of contractible edges of such k-connected graph.

2 Main results

Theorem 3 Let G be a $k(k \geqslant 4)$-connected graph such that any fragment whose neighborhood contains an edge has cardinality exceeding $\frac{k}{4}$, then, G has at least two contractible edges.

Proof Let $\mathcal{S}=\{\{x,y\} \mid xy \in E(G)$ and xy is not contractible $\}$. If $\mathcal{S}=\varnothing$, then Theorem 3 holds. So we may assume that $\mathcal{S} \neq \varnothing$, then we say that a fragment F is an $\mathcal{S}$ fragment if $N(F)$ contains some edges. Let A be an $\mathcal{S}$ fragment with minimum cardinality, then we say A is an $\mathcal{S}$ atom. Clearly, Theorem 3 follows from the following Claim 1. So, in the rest of the paper, our goal is to prove Claim 1.

Claim 1 Let G be a $k(k \geqslant 4)$-connected graph such that any fragment whose neighborhood contains an edge has cardinality exceeding $\frac{k}{4}$, A is an $\mathcal{S}$

atom of G. Then, there are at least two contractible edges which connect A and $N(A)$.

Proof of Claim 1 For otherwise, assume that there is at most one contractible edge join A and $N(A)$.

Lemma 1 Let F_1, F_2 be two $\mathscr{S}$-fragments of G such that at least one of $F_1 \cap F_2$ and $\overline{F_1} \cap \overline{F_2}$ is empty and at least one of $F_1 \cap \overline{F_2}$ and $\overline{F_1} \cap F_2$ is empty. Then

(1) all of $N(F_1) \cap F_2, N(F_2) \cap F_1, N(F_1) \cap \overline{F_2}, N(F_2) \cap \overline{F_1}$ has cardinality more than $\frac{k}{4}$.

(2) $|N(F_2) \cap N(F_1)| < \frac{k}{2}$.

Proof By symmetry, we prove the case $F_1 \cap F_2 = \varnothing$ and $F_1 \cap \overline{F_2} = \varnothing$, the other cases can be proved similarly.

Clearly, $N(F_2) \cap F_1 = F_1$ is an $\mathscr{S}$-fragment, so $|N(F_2) \cap F_1| > \frac{k}{4}$. Now if $F_2 \cap \overline{F_1} = \varnothing$, then $N(F_1) \cap F_2 = F_2$ is an $\mathscr{S}$-fragment, and hence, $|N(F_1) \cap F_2| > \frac{k}{4}$. If $F_2 \cap \overline{F_1} \neq \varnothing$, then we have $|N(F_1) \cap F_2| \geqslant |N(F_2) \cap F_1| > \frac{k}{4}$. Similarly, we can show that $N(F_1) \cap \overline{F_2}, N(F_2) \cap \overline{F_1}$ have cardinality more than $\frac{k}{4}$. Thus (1) holds. Clearly, (1) implies (2), so (2) holds.

Lemma 2 Let F be an $\mathscr{S}$-fragment with $A \cap N(F) \neq \varnothing$ and $(A \cup N(A)) \cap N(F)$ contain an edge, then

(1) $A \subseteq N(F)$.

(2) $|F \cap N(A)| > \frac{k}{4}$, $|\overline{F} \cap N(A)| > \frac{k}{4}$.

Proof Assume that $A \cap F \neq \varnothing$, then, as $A \cap N(F) \neq \varnothing$ and $(A \cup N(A)) \cap N(F)$ contain an edge, we know that $A \cap F$ is not a fragment. It follow that $\overline{A} \cap \overline{F} = \varnothing$ and $|A \cap N(F)| > |N(A) \cap \overline{F}|$. Now we find that $\overline{F}$ is an $\mathscr{S}$-fragments and $|\overline{F}| < |A|$, this contradicts to the choice of A. So we have $A \cap F = \varnothing$, and similarly, $A \cap \overline{F} = \varnothing$. Thus (1) holds and, by Lemma 1, (2) holds.

Lemma 3 A contains some edges.

Proof For otherwise, for any vertex $a \in A$, we find that $N(a) = N(A)$ and there is a fragment which has cardinality 1 but its neighborhood contains an edge, a contradiction.

Lemma 4 For any vertex $b \in N(A)$, $|N(b) \cap A| \geqslant 2$.

Proof First we can show that $|A| \leqslant \frac{k}{2}$. For otherwise, let $|A| > \frac{k}{2}$, then every fragment of G whose neighborhood contain an edge has cardinality more than $\frac{k}{2}$, then, by Theorem 2, the contractible edges of G induce a spanning 2-connected graph. It follows that there are two contractible edges connected A and $N(A)$, a contradiction.

Now assume that $|N(b) \cap A| = 1$ and we will deduce a contradiction. Let $N(b) \cap A = \{a\}$, then we find that $A_1 = A - \{a\}$ is a fragment of G and $|A_1| < |A|$. So by the choice of A, we have $N(A_1)$ contain no edge. This means that $N(a) \subseteq A \cup \{b\}$ and then we can find that $|A| \geqslant k-1 > \frac{k}{2}$ as $d(a) \geqslant k$ and $k \geqslant 4$. This is a contradiction.

Now let $\mathscr{S}_1 = \{\{a,b\} \mid a \in A, b \in N(A), ab \in E(G)$ and ab is not contractible $\}$. By the fact that $k \geqslant 4$ and our assumption, $\mathscr{S}_1 \neq \varnothing$. Then for any $\mathscr{S}_1$ fragment F, we have $A \subseteq N(F)$ by Lemma 2. Let F_1 be a fragment with minimum cardinality under the condition that $A \subseteq N(F_1)$. Let F_2 be a fragment with minimum cardinality under the condition that $A \subseteq N(F_2)$ and $F_2 \subseteq \overline{F_1}$. Clearly, by the fact that $\mathscr{S}_1 \neq \varnothing$, F_1 and F_2 does exist.

Lemma 5 $A \cup F_1 \subseteq N(F)$ for any fragment F of G with $A \subseteq N(F)$ and $N(F) \cap F_1 \neq \varnothing$.

Proof First show that $F_1 \cap F = \varnothing$. For otherwise, assume $F_1 \cap F \neq \varnothing$. Then $|(F_1 \cap N(F)) \cup (N(F_1) \cap N(F)) \cup (F \cap N(F_1))| > k$ and $|F_1 \cap N(F)| > |F \cap N(F_1)|$ (For otherwise, we have $F_1 \cap F$ is a fragment with $A \subseteq N(F_1 \cap F)$, this contradicts to the choice of F_1 as $|F_1 \cap F| < |F_1|$).

It follow that $\overline{F} \cap \overline{F_1} = \varnothing$. Now we find that $|\overline{F}| < |F_1|$, a contradiction. Similarly, we can show that $F_1 \cap \overline{F} = \varnothing$ and thus $F_1 \subseteq N(F)$.

Lemma 6 There exists a vertex $b \in F_2 \cap N(A)$ such that for any fragment F with $A \cup F_1 \subseteq N(F)$, we have $b \notin N(F)$.

Proof Lemma 6 is clearly hold if all fragment

F with $A \cup F_1 \subseteq N(F)$ then $N(F) \cap F_2 = \varnothing$. So assume there exists a fragment F such that $N(F) \cap F_2 \neq \varnothing$.

Claim 2 There is a minimal fragment B with $A \cup F_1 \subseteq N(B)$ and $N(B) \cap F_2 \neq \varnothing$ such that $B \cap F_2 \cap N(A) \neq \varnothing$.

Let B be a minimal fragment with $A \cup F_1 \subseteq N(B)$ and $N(B) \cap F_2 \neq \varnothing$ and B' be such a fragment that is contained in $\overline{B}$. Now assume that $B \cap F_2 \cap N(A) = \varnothing$, $B' \cap F_2 \cap N(A) = \varnothing$ and we try to deduce a contradiction. Now if $|N(B) \cap F_2 \cap N(A)| > \frac{k}{4}$, then we have $|N(A) \cap N(B)| > \frac{k}{2}$. This contradicts to Lemma 1. So we may assume that $|N(B) \cap F_2 \cap N(A)| < \frac{k}{4}$, it follows that $\overline{B} \cap F_2 \cap N(A) \neq \varnothing$ as $|N(A) \cap F_2| \geqslant |A| > \frac{k}{4}$. Now by the fact that $F_1 \subseteq N(B) \cap \overline{F_2}$ and the choice of F_2, we have $\overline{F_2} \cap B = \varnothing$. It follows that $|N(A) \cap B \cap N(F_2)| > \frac{k}{4}$. Similarly, $|N(A) \cap B' \cap N(F_2)| > \frac{k}{4}$, this means that $|N(A) \cap N(F_2)| > \frac{k}{2}$, and contradicts to Lemma 1. Thus Claim 2 holds.

Let B be a minimal fragment with $A \cup F_1 \subseteq N(B)$ and $N(B) \cap F_2 \neq \varnothing$ such that $B \cap F_2 \cap N(A) \neq \varnothing$, and let $b \in B \cap F_2 \cap N(A)$. Now we say that b is just the vertex.

For otherwise, assume that $F_1 \cap A \cup \{b\} \subseteq N(F)$. Now focus on B and F, we can show that either $B \cap F = \varnothing$ or $\overline{B} \cap \overline{F} = \varnothing$ (For otherwise, we can show that $B \cap F$ is a fragment with $A \cup F_1 \subseteq N(B \cap F)$ and $N(B \cap F) \cap F_2 \neq \varnothing$, this contradicts to the choice of B). Similarly, we have either $B \cap \overline{F} = \varnothing$ or $\overline{B} \cap F = \varnothing$. Now we prove the case $B \cap F = \varnothing$ and $B \cap \overline{F} = \varnothing$, the other case can be deduced similarly. As $B \cap F = \varnothing$ and $B \cap \overline{F} = \varnothing$, then we find that $|N(A) \cap B| > \frac{k}{4}$. It follows that $|N(A) \cap F| > \frac{k}{2}$, this contradicts to Lemma 1.

Lemma 7 $A \cup F_2 \subseteq N(F)$ for any fragment F with $A \cup \{b\} \subseteq N(F)$.

Proof By Lemma 6, we have $N(F) \cap F_1 = \varnothing$. So we may assume that $F_1 \subseteq \overline{F}$. It follows that $F_1 \subseteq \overline{F_2} \cap \overline{F}$.

First, we show that $F \cap F_2 = \varnothing$. For otherwise, $F \cap F_2 \neq \varnothing$, then $F \cap F_2$ is a fragment such that $A \subseteq N(F \cap F_2)$ and $F_1 \subseteq \overline{F \cap F_2}$. On the other hand, we have $|F \cap F_2| < |F_2|$, which contradicts to the choices of F_2. So $F \cap F_2 = \varnothing$. Now if $\overline{F} \cap F_2 \neq \varnothing$, then by the choice of F_2, we can find that $F \cap \overline{F_2} = \varnothing$ and $|F| < |F_2|$, which contradicts to the choices of F_2. Thus $\overline{F} \cap F_2 = \varnothing$, $A \cup F_2 \subseteq N(F)$, and Lemma 7 holds.

Let F_3 be a fragment with minimum cardinality under the condition that $A \cup F_1 \subseteq N(F_3)$ and $b \in \overline{F_3}$. Clearly, the number of edges connecting A and F_1 is more than 2. So there is an uncontractible edge which connects A and F_1, we take a smallest separator T contain such an edge, F be a T-fragment. Then by Lemma 5 we have $A \cup F_1 \subseteq T$. Now by Lemma 6, such a fragment does exist.

Let F_4 be a fragment with minimum cardinality under the condition that $A \cup F_2 \subseteq N(F_4)$ and $F_1 \in \overline{F_4}$. By the assumption and Lemma 4, there is an uncontractible edge which connects A and b, we take a smallest separator T contain such an edge, F be a T-fragment. Then, by Lemma 7, we have $A \cup F_2 \subseteq T$, so F_4 does exist.

Notice that F_1, F_2, F_3 and F_4 are all $\mathscr{S}$-fragments.

Lemma 8 (1) $F_4 \cap F_3 = \varnothing$ or $\overline{F_4} \cap \overline{F_3} = \varnothing$.

(2) $F_4 \cap \overline{F_3} = \varnothing$ or $\overline{F_4} \cap F_3 = \varnothing$.

Proof If $F_4 \cap F_3 \neq \varnothing$ and $\overline{F_4} \cap \overline{F_3} \neq \varnothing$, then both $F_4 \cap F_3$ and $\overline{F_4} \cap \overline{F_3}$ are fragments, of which the neighborhood contains A. So we have $|N(A) \cap (F_4 \cap F_3)| > \frac{k}{4}$ and $|N(A) \cap (\overline{F_4} \cap \overline{F_3})| > \frac{k}{4}$. Now a simple calculation shows that $|N(A)| > k$, a contradiction. Thus (1) holds and, then (2) holds similarly.

Lemma 9 $F_2 \subseteq \overline{F_3}$.

Proof For otherwise, we assume that $N(F_3) \cap F_2 \neq \varnothing$, then focus on F_2, F_3, we have $b \in F_2 \cap \overline{F_3}$. It follows that $\overline{F_2} \cap F_3 = \varnothing$, for otherwise, $F_2 \cap \overline{F_3}$ is a fragment such that $A \subseteq N(F_2 \cap \overline{F_3})$, $F_1 \subseteq$

$\overline{F_2 \cap F_3}$ and $|F_2 \cap \overline{F_3}| < |F_2|$, a contradiction. Similarly, we can show that either $F_2 \cap F_3 = \emptyset$ or $\overline{F_2} \cap \overline{F_3} = \emptyset$. By Lemma 1, $|N(F_3) \cap F_2| > \frac{k}{4}$ and hence, $|N(F_3) \cap N(F_4)| \geqslant |A| + |N(F_3) \cap F_2| > \frac{k}{2}$. It is a contradiction by Lemma 8 and Lemma 1.

Now we are ready to complete the proof of Claim 1. First we can assert that $F_4 \cap F_3 \neq \emptyset$. For otherwise, we have $F_4 \cap F_3 = \emptyset$, then $|N(A)| \geqslant |N(A) \cap F_3| + |N(A) \cap F_4| + |N(A) \cap F_1| + |N(A) \cap F_2| > n$, a contradiction. Thus, by Lemma 8, $\overline{F_4} \cap \overline{F_3} = \emptyset$. Now as $F_2 \subseteq \overline{F_1}$, we have $N(F_2) \cap F_1 = \emptyset$, it follow that $|\overline{F_4} \cap N(F_3)| > \frac{k}{2}$. Thus, $|F_4 \cap N(F_3)| < \frac{k}{4}$. So we have $|F_4 \cap N(F_3)| < |\overline{F_3} \cap N(F_4)|$, and hence, $F_4 \cap F_3 = \emptyset$, a contradiction. This contradiction completes the proof of the claim 1.

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(责任编辑:尹　闯)

德国在实验室制造出黑洞等离子体

黑洞的重力很大,会吸附一切物质。进入黑洞后,任何东西都不可能从黑洞的边界之内逃逸出来。随着被吸入的物体的温度不断升高,会产生核与电子分离的高温等离子体。黑洞吸附物质会产生X射线,X射线反过来又会刺激其中的大量化学元素发射出具有独特线条(颜色)的X射线。分析这些线条可以帮助科学家了解更多有关黑洞附近等离子体的密度、速度和组成成分等信息。在宇宙中的储量并不如更轻的氢和氦丰富的铁,能够更好地吸收和重新发射出X射线。铁发射出的光子因此也比其他更轻的原子发射出的光子具有更高的能量、更短的波长(使得其具有不同的颜色)。铁发射出的X射线在穿过黑洞周围的介质时也会被吸收。在这个所谓的光离化过程中,铁原子通常会经历几次电离,其包含的26个电子中有超过一半会被去除,最终产生带电离子,带电离子聚集成为等离子体。德国科研人员在实验室中重现了这个过程。

德国科研人员在实验室中设计一个电子束离子阱。在这个离子阱中,铁原子经由一束强烈的电子束加热,从而被离子化14次。实验过程如下:一团铁离子(仅仅几厘米长并且像头发丝一样薄)在磁场和电场的作用下被悬停在一个超高真空内,同步加速器发射出的X射线的光子能量被一台精确性超高的"单色仪"挑选出来,作为一束很薄但却集中的光束施加到铁离子上。实验室测量到的光谱线与钱德拉X射线天文台和牛顿X射线多镜望远镜所观测的结果相匹配。也就是说,研究人员在地面实验室人为制造出了太空中的黑洞等离子体。

这种新奇的方法将带电离子的离子阱和同步加速器辐射源结合在一起,让人们可以更好地了解黑洞周围的等离子体或者活跃的星系核。应用这种新奇的方法,之前只能在太空由人造卫星执行的天文物理实验也可以在地面进行,诸多天文物理学难题有望得到解决。

(据科学网)