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Positive Solutions of a Singular Fourth Order Boundary Value Problem*

一类四阶奇异边值问题的正解

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Abstract: A boundary value problem for the beam equation $u^{(4)}(t)+g(t)F(t,u(t))=0, 0<t<1$, together with boundary conditions $u(0)=u'(0)=u'(1)=u''(0)=0$, is considered, where $F(t,u)$ may be singular at $u=0$, $g(t)$ may be singular at both ends $t=0$ and $t=1$. By using a Green function of third order two point boundary value problems and Krasnosel'skii fixed point theorem, some sufficient conditions for the existence of at least one positive solution for the boundary value problem are established. An example also is given to illustrate the main results.

Key words: singular boundary value problem, Krasnosel'skii fixed point theorem, positive solution

摘要:利用三阶两点边值问题的格林函数,结合 Krasnosel'skii 不动点定理,考虑梁方程 $u^{(4)}(t)+g(t)F(t,u(t))=0, 0<t<1, u(0)=u'(0)=u'(1)=u''(0)=0$ 的边值问题,其中函数 $F(t,u)$ 在边界 $u=0$ 可能是奇异的,函数 $g(t)$ 在边界 $t=0$ 和 $t=1$ 也可以是奇异的. 获得该系统至少存在一个正解的几组充分条件,并用例子说明主要结果是可行的.

关键词:奇异边值问题 Krasnosel'skii 不动点原理 正解

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Consider the fourth order differential equation

$$u^{(4)}(t)+g(t)F(t,u(t))=0, 0<t<1, \quad (0.1)$$

together with boundary conditions

$$u(0)=u'(0)=u'(1)=u''(0)=0, \quad (0.2)$$

where $F(t,u)\in C([0,1]\times(0,+\infty),[0,+\infty))$ may be singular at $u=0$, $g(t)\in C((0,1),[0,+\infty))$ may be singular at both ends $t=0$ and $t=1$.

Equation (0.1) is often referred as the beam equation. It describes the deflection of a beam under a certain force. The boundary conditions (0.2) mean that the beam is embedded at the end $t=0$, and free at the end $t=1$. Recently, the existence and multiplicity of positive solutions of equation (0.1) in the non-singular case has been extensively studied under various boundary conditions[1~6]. Yao[1] studied the existence of multiple positive solutions for the fourth order boundary value problem

$$u^{(4)}(t)=f(t,u(t),u''(t)), 0\leqslant t\leqslant 1,$$

$$u(0)=u(1)=u''(0)=u''(1)=0,$$

under some semilinear condition. Ma[2] studied posi-

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tive solutions for following boundary value problem

$$u^{(4)}(t) = \lambda f(t, u(t), u'(t)),$$
$$u(0) = u'(0) = u''(1) = u'''(1) = 0,$$

under some superlinear semipositone conditions. For some other results on boundary value problems of the beam equation, we refer the reader to the literatures[3～6].

However for singular fourth order boundary value problems, the research has proceeded very slowly [7～10]. Ma and Tisdell [11] studied the singular sublinear fourth order boundary value problems

$$u^{(4)}(t) = p(t)u^{\lambda}, 0 < t < 1,$$
$$u(0) = u(1) = u'(0) = u'(1) = 0.$$

Following, Cui [12] obtained necessary and sufficient conditions for the existence of position solutions of above problem under superlinear condition.

Inspired by the above mentioned literatures and the literatures[13,14], by using a Green function of third order two point boundary value problems and Krasnosel'skii fixed point theorem of cone expansion compression type, the existence and nonexistence of positive solutions of the problem (0.1)～(0.2) is studied. One must point out that the idea of introducing a Green function of third order two point boundary value problems to study four order boundary value problems is stimulated by the works of Yao[1,13]. However, to the best of the author´s knowledge, there are few authors that have applied such a technique to the singular problem.

1 Preliminaries

Let E be a Banach space, K a cone in E, $K_r = \{u \in K: \| u \| < r\}$, $\partial K_r = \{u \in K: \| u \| = r\}$, $\overline{K}_{r,R} = \{u \in K: r \leqslant \| u \| \leqslant R\}$, where $0 < r < R < +\infty$.

For convenience, one introduces the fixed point theorem as follows, which is due to Krasnosel'skii[15].

Lemma 1.1 Let $(E, \| \cdot \|)$ be a Banach space over the reals, and let $K \subset E$ be a cone in E. Let $0 < r < R$ be real numbers. If $T: \overline{K}_{r,R} \to K$ is a completely continuous operator such that either

(K1) $\| Tu \| \leqslant \| u \|, u \in \partial K_R$, and $\| Tu \| \geqslant \| u \|, u \in \partial K_r$,

or

(K2) $\| Tu \| \geqslant \| u \|, u \in \partial K_R$, and $\| Tu \| \leqslant \| u \|, u \in \partial K_r$.

Then T has a fixed point in $\overline{K}_{r,R}$.

Throughout the rest of the paper, let $E = C[0,1] = \{u \in C[0,1]\}$, and

$$K = \{u \in E: u(t) \geqslant (\frac{1}{3}t^3 - \frac{1}{4}t^4) \| u \| = p(t) \| u \| \},$$

be with norm $\| u \| = \sup_{t \in [0,1]} | u(t) |$. Clearly E is a Banach space, and K is a positive cone in E.

Throughout the paper, one also assumes the following conditions:

(H1) $F: [0,1] \times (0, +\infty) \to [0, +\infty)$ is continuous;

(H2) $g: (0,1) \to [0, +\infty)$ is continuous, $\int_0^1 t(1-t)^2 g(t)\mathrm{d}t < +\infty$;

(H3) $\lim_{n \to +\infty} \int_0^{\frac{1}{n}} t(1-t)^2 g(t) F(t, u(t))\mathrm{d}t = 0$, for all $u(t) \in E$.

Now let $G(t,s)$ be the Green's function of the linear problem

$$\begin{cases} u''' = 0, 0 < t < 1, \\ u(0) = u(1) = u'(0) = 0, \end{cases}$$

which can be explicitly given by

$$G(t,s) = \begin{cases} \frac{1}{2}s^2(1-t)^2 + s(1-t)(t-s), \\ \quad 0 \leqslant t \leqslant s \leqslant 1, \\ \frac{1}{2}t^2(1-s)^2, 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{1.1}$$

It is clear that for all $s, t \in [0,1]$,

$$\sup_{0 \leqslant t \leqslant 1} G(t,s) = \frac{1}{2}s(1-s)^2, \tag{1.2}$$

$$G(t,s) \leqslant \frac{1}{2}t^2(1-t), \tag{1.3}$$

$$G(t,s) \geqslant t^2(1-t) \sup_{0 \leqslant t \leqslant 1} G(t,s). \tag{1.4}$$

Define the operator T by

$$(Tu)(t) = \int_0^t \int_0^1 G(\tau,s) F(s, u(s))\mathrm{d}s\mathrm{d}\tau,$$

$$t \in [0,1]. \tag{1.5}$$

It is easy to verify that a necessary and sufficient condition for the problem (0.1)～(0.2) to have solutions is that $u = Tu$ have fixed point.

Lemma 1.2 $TK \subset K$, and $T\overline{K}_{r,R} \subset K$.

Proof If $u \in K, t \in [0,1]$, then

$$(Tu)(t)=\int_0^t\int_0^1 G(\tau,s)g(s)F(s,u(s))\mathrm{d}s\mathrm{d}\tau\geqslant$$

$$\int_0^t\int_0^1\tau^2(1-\tau)\sup_{\tau\in[0,1]}G(\tau,s)g(s)F(s,u(s))\mathrm{d}s\mathrm{d}\tau\geqslant$$

$$\int_0^t\tau^2(1-\tau)\mathrm{d}\tau\int_0^1\sup_{\tau\in[0,1]}G(\tau,s)g(s)F(s,u(s))\mathrm{d}s=$$

$$p(t)\sup_{\tau\in[0,1]}\int_0^1 G(\tau,s)g(s)F(s,u(s))\mathrm{d}s=$$

$$p(t)\int_0^1\sup_{\tau\in[0,1]}\int_0^1 G(\tau,s)g(s)F(s,u(s))\mathrm{d}s\mathrm{d}\tau\geqslant$$

$$p(t)\int_0^1\int_0^1 G(\tau,s)g(s)F(s,u(s))\mathrm{d}s\mathrm{d}\tau=$$

$$p(t)\sup_{t\in[0,1]}\int_0^t\int_0^1 G(\tau,s)g(s)F(s,u(s))\mathrm{d}s\mathrm{d}\tau=$$

$$p(t)\|Tu\|.$$

Thus, $TK\subset K$, i.e. $T\overline{K}_{r,R}\subset K$. The proof is complete.

Lemma 1.3 Suppose that (H1), (H2) and (H3) hold. Then $T:\overline{K}_{r,R}\to K$ is a completely continuous operator.

Proof First, for any $r>0$, $\sup_{\partial K_r}\int_0^t\int_0^1 G(\tau,s)g(s)F(s,u(s))\mathrm{d}s\mathrm{d}\tau<+\infty$ for all t holds. Thus $T: K\backslash\{0\}\to C[0,1]$ is well defined.

In fact, $p(t)r\leqslant u(t)\leqslant r$. It follows from conditions (H2) and (H3) that there exists N such that $\int_0^{\frac{1}{N}}t(1-t)^2g(t)F(t,u(t))\mathrm{d}t<1$, for $t<\frac{1}{N}$.

$$(Tu)(t)=\int_0^t\int_0^1 G(\tau,s)g(s)F(s,u(s))\mathrm{d}s\mathrm{d}\tau\leqslant\int_0^1\int_0^1 G(\tau,s)g(s)F(s,u(s))\mathrm{d}s\mathrm{d}\tau\leqslant\int_0^1\int_0^1\frac{1}{2}s(1-s)^2g(s)F(s,u(s))\mathrm{d}s\mathrm{d}\tau=\int_0^{\frac{1}{N}}\frac{1}{2}s(1-s)^2g(s)F(s,u(s))\mathrm{d}s+\int_{\frac{1}{N}}^1\frac{1}{2}s(1-s)^2g(s)F(s,u(s))\mathrm{d}s<1+\int_{\frac{1}{N}}^1\frac{1}{2}s(1-s)^2g(s)F(s,u(s))\mathrm{d}s.\quad(1.6)$$

Let $L=\sup_{t\in[\frac{1}{N},1]}\{F(t,u(t)):p(t)r\leqslant u(t)\leqslant r\}$,

$$(Tu)(t)\leqslant 1+L\int_{\frac{1}{N}}^1\frac{1}{2}s(1-s)^2g(s)\mathrm{d}s<+\infty.\quad(1.7)$$

Second, $T:\overline{K}_{r,R}\to K$ is compact and continuous. Assume that $u_n,u_0\in\overline{K}_{r,R}$ and $\|u_n-u_0\|\to 0$, for $n\to\infty$. Obviously, $p(t)r\leqslant u_n,u_0\leqslant R, n=1,2,\cdots$. In view of $|F(t,u_n)-F(t,u_0)|\to 0, n\to\infty$ for any $t\in[0,1]$, and from Lebesgue Control Converges Theorem, one has

$$\|Tu_n-Tu_0\|=\|\int_0^t\int_0^1 G(\tau,s)g(s)[F(s,u_n(s))-F(s,u_0(s))]\mathrm{d}s\mathrm{d}\tau\|\leqslant\int_0^1\frac{1}{2}s(1-s)^2g(s)|F(s,u_n(s))-F(s,u_0(s))|\mathrm{d}s\to 0, n\to\infty.$$

Thus, $T:\overline{K}_{r,R}\to K$ is continuous.

Let $D\subset\overline{K}_{r,R}$ be bounded, i.e., $\|u\|\leqslant M$ for all $u\in D$ and some $M>0$. It is clear that $u\in D$ satisfies $u\in\overline{K}_{r,R}$. Thus, it is easy to prove that there is a constant such that $|Tu(t)|\leqslant L_1$.

$$|(Tu)'(t)|=|\int_0^1 G(t,s)g(s)F(s,u(s))\mathrm{d}s|\leqslant\int_0^1\sup_{0\leqslant t\leqslant 1}G(t,s)g(s)F(s,u(s))\mathrm{d}s\leqslant\int_0^1\frac{1}{2}s(1-s)^2g(s)F(s,u(s))\mathrm{d}s.$$

Similarly, from condition (H3), there is a constant L_2 such that $|(Tu)'(t)|\leqslant L_2$.

So $T(D)$ is uniformly bounded and equicontinuous. From the Ascoli-Arzela Theorem, $T(D)$ is relatively compact. This completes the proof that T is compact.

To sum up, one has proved $T:\overline{K}_{r,R}\to K$ is completely continuous. The proof is complete.

2 Main results

First, one defines some important constants:

$$\limsup_{u\to 0^+}\sup_{t\in[0,1]}\frac{F(t,u)}{u}=F_0,$$

$$\limsup_{u\to+\infty}\sup_{t\in[0,1]}\frac{F(t,u)}{u}=F_\infty,\ \liminf_{u\to 0^+}\inf_{t\in[0,1]}\frac{F(t,u)}{u}=f_0,$$

$$\liminf_{u\to+\infty}\inf_{t\in[0,1]}\frac{F(t,u)}{u}=f_\infty,\ \frac{1}{12}\int_0^1\sup_{t\in[0,1]}G(t,s)g(s)p(s)\mathrm{d}s=A,\ \frac{1}{2}\int_0^1 s(1-s)^2g(s)\mathrm{d}s=B.$$

From the above analysis, one knows, for all $u\in\overline{K}_{r,R}$, the fixed point of the equation (1.3) is the solution of the problem (0.1)~(0.2). Next one will look for the fixed point.

Theorem 2.1 Suppose that (H1), (H2) and (H3) hold. If $BF_0<1<Af_\infty$, the problem (0.1)~(0.2) has at least one positive solution.

Proof First, one chooses $\varepsilon>0$ such that $(F_0+\varepsilon)B\leqslant 1$. From the definition of F_0 one sees that there exists $H_1>0$ such that

$F(t,u)\leqslant(F_0+\varepsilon)u$ for all $t\in[0,1]$, $u\in(0,H_1]$.

For each $u\in K$ with $\|u\|=H_1$, one has

$$\| Tu \| = \int_0^1\int_0^1 G(t,s)g(s)F(s,u(s))\mathrm{d}s\mathrm{d}t \leqslant$$

$$\int_0^1\int_0^1 G(t,s)g(s)(F_0+\varepsilon)u(s)\mathrm{d}s\mathrm{d}t \leqslant (F_0+\varepsilon)\| u \| \cdot$$

$$\int_0^1 \frac{1}{2}s(1-s)^2 g(s)\mathrm{d}s = B(F_0+\varepsilon)\| u \| \leqslant \| u \| .$$

which means $\| Tu \| \leqslant \| u \|$ for $u \in \partial K_{H_1} = \{u \in K : \| u \| = H_1\}$.

To construct K_{H_2}, one chooses $\delta > 0$ and $c \in (0,\frac{1}{4})$ such that $\frac{1}{12}\int_c^1 \sup_{t\in[0,1]} G(t,s)g(s)p(s)\mathrm{d}s(f_\infty - \delta) \geqslant 1$.

There exists $H_3 > 0$ such that

$F(t,u) \geqslant (f_\infty - \delta)u$ for all $t \in [0,1], u \in [H_3, +\infty)$.

Let $H_2 = \max\{12H_3c^{-4}, 2H_1\}$. If $u \in K$ such that $\| u \| = H_2$, for each $t \in [c,1]$, one has

$$u(t) \geqslant H_2 p(t) = H_2(\frac{1}{3}t^3 - \frac{1}{4}t^4) \geqslant H_2 \frac{1}{12}t^4 \geqslant H_2 \frac{1}{12}c^4 = H_3.$$

Therefore, for each $u \in K$ with $\| u \| = H_2$, one has

$$\| Tu \| = \int_0^1\int_0^1 G(t,s)g(s)F(s,u(s))\mathrm{d}s\mathrm{d}t \geqslant$$

$$\int_0^1\int_0^1 t^2(1-t)\sup_{t\in[0,1]} G(t,s)g(s)F(s,u(s))\mathrm{d}s\mathrm{d}t =$$

$$\frac{1}{12}\int_0^1 \sup_{t\in[0,1]} G(t,s)g(s)F(s,u(s))\mathrm{d}s \geqslant$$

$$\frac{1}{12}\int_0^1 \sup_{t\in[0,1]} G(t,s)g(s)u(s)\mathrm{d}s(f_\infty - \delta) \geqslant$$

$$\frac{1}{12}\int_c^1 \sup_{t\in[c,1]} G(t,s)g(s)p(s)\mathrm{d}s(f_\infty - \delta)\| u \| \geqslant$$

$$\| u \| ,$$

which means $\| Tu \| \geqslant \| u \|$ for $u \in \partial K_{H_2} = \{u \in K : \| u \| = H_2\}$, and $H_2 > H_1$. Now that the condition (K2) of Lemma 1.1 is satisfied, there exists a fixed point of T in $\overline{K}_{H_1,H_2}$. The proof is completed.

Theorem 2.2 Suppose that (H1), (H2) and (H3) hold. If $BF_\infty < 1 < Af_0$, the problem (0.1)~(0.2) has at least one positive solution.

Proof First, one chooses $\varepsilon > 0$ such that $(f_0 - \varepsilon)A \geqslant 1$. There exists $H_1 > 0$ such that

$F(t,u) \geqslant (f_0 - \varepsilon)u$ for all $t \in [0,1], u \in (0, H_1]$.

For each $u \in K$ with $\| u \| = H_1$, one has

$$\| Tu \| = \int_0^1\int_0^1 G(t,s)g(s)F(s,u(s))\mathrm{d}s\mathrm{d}t \geqslant$$

$$\int_0^1\int_0^1 t^2(1-t)\sup_{t\in[0,1]} G(t,s)g(s)(f_0-\varepsilon)u(s)\mathrm{d}s\mathrm{d}t \geqslant$$

$$(f_0-\varepsilon)\| u \| \int_0^1 \frac{1}{12}\sup_{t\in[0,1]} G(t,s)g(s)p(s)\mathrm{d}s = A(f_0 - \varepsilon)\| u \| \geqslant \| u \| ,$$

which means $\| Tu \| \geqslant \| u \|$ for $u \in \partial K_{H_1} = \{u \in K : \| u \| = H_1\}$.

To construct K_{H_2}, one chooses $0 < \delta < 1$ such that $(F_\infty + \delta)B \leqslant 1$. There exists $H_3 \geqslant 2H_1 > 0$ such that

$F(t,u) \leqslant (F_\infty + \delta)u$ for all $t \in [0,1], u \in [H_3, +\infty)$.

Obviously, there exists $t' \in [0,1]$ and $H_2 \geqslant H_3$ such that

$F(t,u) \leqslant F(t',H_2) \leqslant (F_\infty + \delta)H_2$, for all $t \in [0,1], u \in [0,H_2]$.

For $u \in K$ with $\| u \| = H_2$, one has

$$u(t) \geqslant H_2 p(t) = H_2(\frac{1}{3}t^3 - \frac{1}{4}t^4) \geqslant H_2 \frac{1}{12}t^4 \geqslant H_2 \frac{1}{12}c^4 = H_3.$$

Therefore, for each $u \in K$ with $\| u \| = H_2$, one has

$$\| Tu \| = \int_0^1\int_0^1 G(t,s)g(s)F(s,u(s))\mathrm{d}s\mathrm{d}t \leqslant$$

$$\int_0^1\int_0^1 \frac{1}{2}s(1-s)^2 g(s)F(s,u(s))\mathrm{d}s\mathrm{d}t =$$

$$\int_0^1 \frac{1}{2}s(1-s)^2 g(s)F(s,u(s))\mathrm{d}s \leqslant (F_\infty+\delta)\int_0^1 \frac{1}{2}s(1-s)^2 g(s)\mathrm{d}sH_2 \leqslant H_2 = \| u \| ,$$

which means $\| Tu \| \leqslant \| u \|$ for $u \in \partial K_{H_2} = \{u \in K : \| u \| = H_2\}$, and $H_2 > H_1$. Now that the condition (K1) of Lemma 1.1 is satisfied, there exists a fixed point of T in $\overline{K}_{H_1,H_2}$. The proof is complete.

Theorem 2.3 Suppose that (H1), (H2) and (H3) hold. If $BF(t,u) < u$, for $t \in [0,1], u \in (0, +\infty)$, the problem (0.1)~(0.2) has no positive solutions.

Proof Assume $u(t)$ is a positive solution of the problem (0.1) and (0.2). Then $u(t) > 0$ for $0 < t \leqslant 1$, and

$$\| u \| = \int_0^1\int_0^1 G(t,s)g(s)F(s,u(s))\mathrm{d}s\mathrm{d}t \leqslant$$

$$\int_0^1 \frac{1}{2}s(1-s)^2 g(s)F(s,u(s))\mathrm{d}s\mathrm{d}t < [\int_0^1 \frac{1}{2}s(1-s)^2 g(s)\mathrm{d}s]^{-1}\int_0^1 \frac{1}{2}s(1-s)^2 g(s)u(s)\mathrm{d}s \leqslant$$

$$B^{-1}\int_0^1 \frac{1}{2}s(1-s)^2 g(s)\mathrm{d}s \| u \| = \| u \| ,$$

which is a contradiction. The proof is complete.

Theorem 2.4 Suppose that (H1), (H2) and (H3) hold. If $AF(t,u) > u$, for $t \in [0,1]$, $u \in (0,+\infty)$, the problem (0.1)~(0.2) has no positive solutions.

The proof of Theorem 2.4 is quite similar to that of Theorem 2.3.

3 Example

Example 3.1 Consider the problem

$$u^{(4)}(t) + g(t)F(t,u(t)) = 0, 0 < t < 1, \quad (3.1)$$

$$u(0) = u'(0) = u'(1) = u''(0) = 0, \quad (3.2)$$

where

$$g(t) = \frac{1}{t(1-t)^2}, 0 < t < 1, \quad (3.3)$$

$$F(t,u(t)) = \lambda \frac{u(1+t+5u)}{5+u}, u > 0, \quad (3.4)$$

and $\lambda > 0$ is a parameter.

It is easy to see that $F_0 = 0.4\lambda$, $F_\infty = f_\infty = 5\lambda$, $f_0 = 0.2\lambda$, and

$0.2\lambda\, u < F(t,u) < 5\lambda\, u$, for all $t \in [0,1]$, $u > 0$.

It is also easy to verify, by direct calculation, that $B = 0.5$, $A > 0.0781$. From Theorem 2.1 we see that if $2.5609 \approx \frac{1}{5A} < \lambda < \frac{5}{2B} = 5$, then the problem (3.1)~(3.4) has at least one positive solution. From Theorem 2.3 we see that if $\lambda \leqslant \frac{1}{5B} = 0.4$, then the problem(3.1)~(3.4) has no positive solutions. From Theorem 2.4 we see that if $\lambda \geqslant \frac{5}{A} \approx 6.4021$, then the problem (3.1)~(3.4) has no positive solutions.

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