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Upper and Lower Bounds for Generalized Multigraph Ramsey Numbers Based on Turán Numbers*

基于 Turán 数的广义多图 Ramsey 数上下界

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Abstract: Multigraph Ramsey numbers are generalized to generalized multigraph Ramsey numbers. Upper bounds and constructive lower bounds for some multigraph Ramsey numbers are obtained based on the Turán numbers for complete graphs, by which their values are obtained.
Key words: multigraph, Ramsey number, Turán number
摘要:将多图 Ramsey 数推广为广义多图 Ramsey 数.利用完全图的 Turán 数,给出一些多图 Ramsey 数的上界和构造性下界,进而确定出它们的准确值.
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The multigraph Ramsey number is defined in reference[1] based on set-coloring, as a generalization of the classic Ramsey number. In this paper, the multigraph Ramsey number $f^{(r)}(a_1, \cdots, a_k)$ is generalized to the generalized multigraph Ramsey number $f^{(r)}(G_1, \cdots, G_k)$, and the upper bounds and constructive lower bounds for some multigraph Ramsey numbers are obtained based on the Turán numbers for complete graphs, by which their values are obtained.

1 Upper bounds for generalized multigraph Ramsey numbers

Let $M_n^{(r)}$ be the multigraph of order n, in which there are r edges between any two different vertices. In reference[1], the multigraph Ramsey number $f^{(r)}(a_1, \cdots, a_k)$ is defined to be the minimum positive integer n such that in any k-edge coloring of $M_n^{(r)}$ (every edge is colored with one among k given colors, and edges between the same pair of vertices are colored with different colors), there must be some $i \in \{1, \cdots, k\}$ such that $M_n^{(r)}$ has a complete subgraph of order a_i, of which all the edges are in color i.

Note that the set-coloring of edges in a graph is the same as the edge-coloring of a multigraph, in which the edges between the same pair of vertices must be in different colors. Multigraph Ramsey number is a generalization of the classic Ramsey number, because that the classical Ramsey number $R(a_1, \cdots, a_k) = f^{(1)}(a_1, \cdots, a_k)$. In reference[2], Ramsey numbers for multigraph were studied without success, because the authors had not used set-coloring.

For simple graph G, H is G-free if H contains no subgraph that is isomorphic to G. Other notations used in this paper can be found in reference[1] and reference[3].

The generalized multigraph Ramsey number $f^{(r)}(G_1, \cdots, G_k)$ is defined to be the minimum positive integer n such that in any k-edge coloring of $M_n^{(r)}$ (every edge is colored with one among k given colors, and edges between the same pair of vertices are colored with different colors), there must be some $i \in \{1, \cdots, k\}$ such that $M_n^{(r)}$ has a subgraph isomorphic to G_i, of which all the edges are in color i.

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If G_i is isomorphic to G for any $i \in \{1, \cdots, k\}$, then we denote $f^{(r)}(G_1, \cdots, G_k)$ by $f_k^{(r)}(G)$.

Some results on multigraph Ramsey numbers can be generalized, and can be proved similarly.

Let Turán number $t(G, n)$ denote the maximum number of edges in any G-free graph of order n. Turán number for complete number is determined in Turán theorem[4].

Theorem 1 If $\sum_{i=1}^{k} t(G_i, n) < r C_n^2$, then

$f^{(r)}(G_1, \cdots, G_k) \leqslant n$.

Proof If $f^{(r)}(G_1, \cdots, G_k) > n$, then there are k subgraphs of K_n, say $H_1, \cdots, H_k$, of order n such that H_i is G_i-free for any $i \in \{1, \cdots, k\}$, and every edge of K_n is in r subgraphs among $\{H_1, \cdots, H_k\}$. Thus

$$\sum_{i=1}^{k} |E(H_i)| = r C_n^2.$$

On the other hand, $E(H_i) \leqslant t(G_i, n)$ because that H_i is G_i-free, which contradicts with $\sum_{i=1}^{k} t(G_i, n) < rC_n^2$. Thus

$f^{(r)}(G_1, \cdots, G_k) \leqslant n$.

In particularly, Theorem 1 together with Turán Theorem can be used to obtain the upper bounds for multigraph Ramsey numbers. Note this approach can not be used to study the upper bound for the classical Ramsey number. For instance, through some simple computation, we can obtain the following upper bounds.

Corollary 1 $f_{C_{2m}^m}^{(\frac{m}{2m-1}C_{2m}^m)}(3) \leqslant 2m+1$ and

$f_{C_{2m+1}^m}^{(\frac{m+1}{2m+1}C_{2m+1}^m)}(3) \leqslant 2m+2$.

Proof By Turán Theorem we know that $t(K_3, 2m+1) = m(m+1)$. Since that

$$\frac{m}{2m-1}C_{2m}^m C_{2m+1}^2 = \frac{(2m+1)m^2}{2m-1}C_{2m}^m > m(m+1)C_{2m}^m,$$

by Theorem 1 we have $f_{C_{2m}^m}^{(\frac{m}{2m-1}C_{2m}^m)}(3) \leqslant 2m+1$. $f_{C_{2m+1}^m}^{(\frac{m+1}{2m+1}C_{2m+1}^m)}(3) \leqslant 2m+2$ can be proved by Theorem 1 similarly.

2 Constructive lower bounds based on Turán graphs

In this section, we will give constructive lower bounds for some generalized multigraph Ramsey numbers based on Turán graphs. Note in reference [1] it was proved that $f_k^{(r)}(q) \leqslant f_{sk}^{(sr)}(q)$.

2.1 Lower bounds for generalized multigraph Ramsey numbers

Theorem 2 If G is a simple graph of order n, and $\mathrm{Aut}(G)$ is the automorphism group of graph G, then

$f_{n!/|\mathrm{Aut}(G)|}^{(2(n-2)!t(G,n)/|\mathrm{Aut}(G)|)}(G) > n$.

Proof Let H be the Turán graph of G, with n vertices and $t(G, n)$ edges. So we can find $n!$ subgraphs that are isomorphic to H in the complete graph K_n. Based on these $n!$ subgraphs of K_n, we can give $M_n^{(r)}$ an $n!$-edge coloring that shows $f_{n!}^{(r)}(G) > n$, where $r = \frac{n!t(G,n)}{C_n^2} = 2(n-2)!t(G,n)$. Thus we obtain that $f_{n!}^{(2(n-2)!t(G,n))}(G) > n$. Similarly, for the $n!$-edge coloring of $M_n^{(r)}$, there are $|\mathrm{Aut}(G)|$ subgraphs isomorphic to H with the edges with common ends, we delete $|\mathrm{Aut}(G)|-1$ ones of them. This $n!/|\mathrm{Aut}(G)|$-edge coloring of $M_n^{(r/|\mathrm{Aut}(G)|)}$ shows that

$f_{n!/|\mathrm{Aut}(G)|}^{(2(n-2)!t(G,n)/|\mathrm{Aut}(G)|)}(G) > n$.

2.2 Values of some multigraph Ramsey number

In this subsection we will give constructive lower bounds for some multigraph Ramsey numbers based on Turán graph.

Lemma 1 $f_{C_{2m}^m}^{(\frac{m}{2m-1}C_{2m}^m)}(3) > 2m$, and

$f_{C_{2m+1}^m}^{(\frac{m+1}{2m+1}C_{2m+1}^m)}(3) > 2m+1$.

Proof Let H be the Turán graph of graph K_3, with $2m$ vertices. By Turán theorem we know that H must be isomorphic to $K_{m,m}$ and of m^2 edges. Therefore similar to the proof of Theorem 2, we have that $f_{C_{2m}^m}^{(\frac{m}{2m-1}C_{2m}^m)}(3) > 2m$. Similarly we can obtain that

$f_{\binom{m}{2m+1}}^{(\frac{m+1}{2m+1}\binom{m}{2m+1})}(3) > 2m+1$.

By Corollary 1 and Lemma 1 we can obtain the following theorem on the value of some multigraph Ramsey numbers.

Theorem 3 $f_{C_{2m}^m}^{(\frac{m}{2m-1}C_{2m}^m)}(3) = 2m+1$, and

$f_{C_{2m+1}^m}^{(\frac{m+1}{2m+1}C_{2m+1}^m)}(3) = 2m+2$.

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