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Limit Cycles Bifurcated from a Z_3 -equivariant Near-Hamiltonian System

Z_3 等变近 Hamiltonian 系统的极限环分支

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Abstract:: The number of limit cycles of a Z_3-equivariant cubic Hamiltonian system under Z_3-equivariant quartic perturbations was studied using the methods of Hopf bifurcation theory. The results show that the perturbed system can have 6 small limit cycles.

Key words: limit cycle, near-Hamiltonian system, Z_3-equivariance, Hopf bifurcation

摘要: 在 Z_3 等变四次扰动下,利用 Hopf 分支理论的方法,证明 Z_3 等变 Hamiltonian 系统可以扰动出 6 个小振幅极限环.

关键词: 极限环 近 Hamiltonian 系统 Z_3 等变 Hopf 分支
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It is well known that the second part of Hilbert 16th problem concerns the maximal number and relative position of limit cycles of the planar polynomial vector fields. There have been many studies on obtaining more limit cycles and various configuration patterns of their relative dispositions. The problem is so hard that it has not been solved completely. To reduce the difficulty one can study the systems with some symmetry. An important symmetry is the Z_q -equivariance which was first introduced in reference [1]. From then on, Yu et al[2] proved that a cubic Z_5-equivariant system can have 1 limit cycle. Yu et al[3] proved that a cubic Z_3-equivariant system can have 3 small limit cycles and 1 big limit cycle. Zhang et al[4] found a quartic system have at least 15 limit cycles.

Using the Hopf bifurcation method, we study the bifurcation of limit cycles in the following cubic Z_3-quintic Hamiltonian system perturbed by quartic Z_3-equivariant polynomials

$$\dot{x} = H_y + \varepsilon P_4(x,y), \dot{y} = -H_x + \varepsilon Q_4(x,y), \quad (0.1)$$

where ε is nonnegative and small, and find the perturbed system can have 6 small limit cycles.

1 Preliminary definitions and lemmas

When $\varepsilon = 0$, formula(0.1) is a Hamiltonian system with Hamiltonian function

$$H(x,y) = xy^2 + \frac{1}{2}x^2y^2 + \frac{1}{4}y^4 - \frac{1}{3}x^3 + \frac{1}{4}x^4.$$

$(P_4(x,y), Q_4(x,y))$ is the quartic polynomial vector invariant under rotation of $\frac{2\pi}{3}$ with respect to the origin O. From reference [1] we know that $(P_4(x,y), Q_4(x,y))$ is respectively the real and imaginary part of the following complex function

$$F(z,\bar{z}) = (A_0 + A_1|z|^2)z + (A_2 + A_3|z|^2)\bar{z}^2 + A_4z^4,$$

where $A_k = a_k + ib_k, k = 0,1,2,\cdots,5, z = x + iy, \bar{z} = x - iy$. It is direct that

$$P_4 = a_0x + (xy^2 + x^3)a_1 + (x^2 - y^2)a_2 + (-y^4 + x^4)a_3 + (-6x^2y^2 + y^4 + x^4)a_4 - b_0y + (-x^2y - y^3)b_1 + 2b_2xy + (2x^3y + 2xy^3)b_3 +$$

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$(4xy^3 - 4x^3y) b_4$,

$Q_4 = a_0 y + (y^3 + x^2 y) a_1 - 2a_2 xy + (-2x^3 y - 2xy^3) a_3 + (-4xy^3 + 4x^3 y) a_4 + b_0 x + (xy^2 + x^3) b_1 + (x^2 - y^2) b_2 + (-y^4 + x^4) b_3 + (-6x^2y^2 + y^4 + x^4) b_4$.

Consider the following near-Hamiltonian system

$$\dot{x} = H_y(x,y) + \varepsilon p(x,y,\delta), \quad \dot{y} = -H_x(x,y) + \varepsilon q(x,y,\delta), \tag{1.1}$$

where $H(x,y)$, $p(x,y,\delta)$ and $q(x,y,\delta)$ are analytic functions, $\varepsilon \geqslant 0$ is small and $\delta \in D \subset R^m$ is a vector parameter with D compact. When $\varepsilon = 0$, system (1.1) becomes a Hamiltonian system of the form

$$\dot{x} = H_y, \dot{y} = -H_x. \tag{1.2}$$

Suppose that system (1.2) has a homoclinic loop L_0 defined by the equation $H(x,y) = \beta$. Then there exists an open interval (α,β) with $h = \beta$ as its endpoint, such that the equation $H(x,y) = h$ for $h \in (\alpha,\beta)$ defines a family of periodic orbits L_h of system (1.2). Further, suppose that the limit of L_h as $h \to \alpha (h \in (\alpha,\beta))$ is a center $C(x_c,y_c)$, denoted by $L_\alpha = (x_c,y_c)$. Then $\alpha = H(x_c, y_c)$. Let

$$M(h,\delta) = \oint_{L_h} q\mathrm{d}x - p\mathrm{d}y, h \in (\alpha,\beta), \tag{1.3}$$

which is called a Melnikov function. Let

$$p(x,y,\delta) = \sum_{i+j\geqslant 0} a_{ij} x^i y^j, q(x,y,\delta) = \sum_{i+j\geqslant 0} b_{ij} x^i y^j. \tag{1.4}$$

Then from references [5,6], we have

Lemma 1.1[5] Let $M(h,\delta)$ be given by formula (1.3). Then

$$M(h,\delta) = c_0 + O(1),$$

for $0 < |h - \beta| \ll 1$ and $h \in (\alpha,\beta)$ near the homoclinic loop L_1, where

$$c_0(\delta) = M(0,\delta) = \int_{L_1} q\mathrm{d}x - p\mathrm{d}y |_{\varepsilon=0}. \tag{1.5}$$

Lemma 1.2[6] (i) The Melnikov function $M(h, \delta)$ has the form

$$M(h,\delta) = \sum_{k\geqslant 0} b_k (h-\alpha)^{k+1} \text{ for } 0 < |h-\alpha| \ll 1, h \in (\alpha,\beta), \tag{1.6}$$

near the center $L_\alpha = (x_c,y_c)$, where b_k is the function of the relevant coefficients. (ii) Further suppose

$$H(x,y) = \alpha + \frac{1}{2}((x-x_c)^2 + (y-y_c)^2) + \sum_{i+j\geqslant 3} h_{ij}(x-x_c)^i (y-y_c)^j,$$

$$p_x + q_y = \sum_{i+j\geqslant 0} c_{ij}(x-x_c)^i (y-y_c)^j.$$

Then

$$B_0 = 2\pi c_{00}, B_1 |_{B_0=0} = -c_{10}\pi(h_{12} + 3h_{30}) - c_{01}\pi(h_{21} + 3h_{03}) + c_{20}\pi + c_{02}\pi,$$

$$B_2 |_{B_0=0} = c_{10}A_{10} + c_1 A_{01} + c_{20}A_{20} + c_{11}A_{11} + c_{02}A_{02} - c_{30}\pi(h_{12} + 5h_{30}) - c_{12}\pi(h_{30} + h_{12}) - c_{21}\pi(h_3 + h_{21}) - c_{03}\pi(h_{21} + 5h_3) + \pi c_{40} + \pi c_4 + \frac{1}{3}\pi c_{22},$$

with

$$A_{10} = 35\pi h_{30}h_{40} + 5\pi(h_{12}h_{04} + h_{12}h_{40} + h_{21}h_{31} + h_{03}h_{13} + h_{30}h_{22}) + 3\pi(h_{12}h_{22} + h_{13}h_{21} + h_{03}h_{31} + h_{04}h_{30}) - \frac{5}{2}\pi(h_{12}^3 + 3h_{03}^2h_{30} + 3h_{12}^2h_{30} + 3h_{21}^2h_{12} + 6h_{12}h_{21}h_{03} + 6h_{03}h_{21}h_{30}) - \pi(h_{14} + h_{32} + 5h_{50}) - \frac{105}{2}\pi h_{30}^3 - \frac{35}{2}\pi(h_{03}^2h_{12} + h_{21}^2h_{30} + h_{30}^2h_{12}),$$

$$A_{01} = 35\pi h_{03}h_{04} + 5\pi(h_{21}h_{40} + h_{21}h_{04} + h_{12}h_{13} + h_{30}h_{31} + h_{03}h_{22}) + 3\pi(h_{21}h_{22} + h_{12}h_{31} + h_{30}h_{13} + h_{40}h_{03}) - \frac{5}{2}\pi(h_{21}^3 + 3h_{30}^2h_{03} + 3h_{21}^2h_{03} + 3h_{12}^2h_{21} + 6h_{12}h_{21}h_{30} + 6h_{03}h_{12}h_{30}) - \pi(h_{41} + h_{23} + 5h_{05}) - \frac{105}{2}\pi h_{03}^3 - \frac{35}{2}\pi(h_{30}^2h_{21} + h_{12}^2h_{03} + h_{03}^2h_{21}),$$

$$A_{20} = \frac{35}{2}\pi h_{30}^2 + \frac{5}{2}\pi(h_{03}^2 + h_{21}^2 + 2h_{12}h_{30}) + \frac{3}{2}\pi(h_{12}^2 + 2h_{03}h_{21}) - \pi(h_{04} + h_{22} + 5h_{40}),$$

$$A_{11} = 5\pi(h_{03}h_{12} + h_{30}h_{21}) + 3\pi(h_{03}h_{30} + h_{12}h_{21}) - \pi(h_{13} + h_{31}),$$

$$A_{02} = \frac{35}{2}\pi h_{03}^2 + \frac{5}{2}\pi(h_{30}^2 + h_{12}^2 + 2h_{21}h_{03}) + \frac{3}{2}\pi(h_{21}^2 + 2h_{30}h_{12}) - \pi(h_{40} + h_{22} + 5h_{04}).$$

Following the idea used in reference [7], we have immediately the following Lemma, of which the proof is not difficulty and we omit it.

Lemma 1.3 Let formula (1.5) and formula (1.6) hold, suppose there exists $\delta_0 \in D$ such that

$$B_0(\delta_0) = B_1(\delta_0) = 0, B_2(\delta_0) \neq 0, c_0(\delta_0) \neq 0,$$

and

$$\operatorname{rank} \frac{\partial(B_0,B_1)}{\partial(\delta_1,\delta_2,\cdots,\delta_m)} |_{\delta=\delta_0} = 2.$$

Then there exist some (ε,δ) near $(0,\delta_0)$ such that system (1.1) have $2 + \frac{1 - \operatorname{sgn}(c_0(\delta_0)B_2(\delta_0))}{2}$ limit cycles in the homoclinic loop L_1, two of which are near the center.

2 Main result

Theorem 2.1 Taking all $b_i = 0, i = 0,1,\cdots,4$, there exist some $\{a_0,a_1,\cdots,a_5\}$ such that the system (0.1) has 6 small limit cycles for ε positive and very small, and the configuration of these limit cycles is shown in the Figure 1.

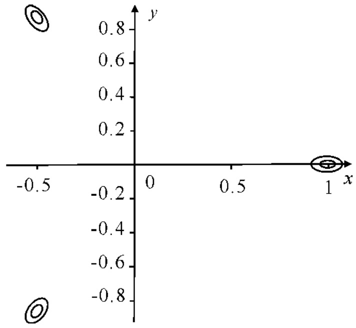

Fig. 1 Distribution of 28 limit cycles of formula(0.1)

Proof We first study the portrait of the Z_3-equivariant system (0.1) with $\varepsilon = 0$. The Z_3-equivariant system (0.1) with $\varepsilon = 0$ has a compound cycle denoted by Γ consisting of 3 homoclinic loop L_1, L_2, L_3, defined by $H(x,y) = 0$, and there is a center C_i in every L_i, defined by $H = -\frac{1}{12}$, where $C_1(1,0)$, $C_2(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $C_2(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$, then we have

$$M_i(h,\delta) = \oint_{L_h} q_4 \mathrm{d}x - p_4 \mathrm{d}y, h \in (-\frac{1}{12}, 0), i = 1, 2, 3,$$

where $\delta = (a_0, a_1, \cdots, a_5, b_0, \cdots, b_5) \in R^{12}$. By Z_3-equivariance, $M_1(h,\delta) = M_2(h,\delta) = M_3(h,\delta)$, we can only study $M_1(h,\delta)$. By Lemma 1.1, we have

$$M(h,\delta) = c_0 + O(1),$$

for $0 < -h \ll 1$ near the homoclinic loop L_1.

By formula(1.5), with the help Maple 13, we can directly have

$$c_0 = \frac{8}{27}\pi a_0 + \frac{32}{81}\pi a_1 + \frac{128}{729}\pi a_3 + \frac{512}{729}\pi a_4. \tag{2.1}$$

So we have

$$M_1(h,\delta) = \sum_{k \geq 0} b_k (h + \frac{1}{12})^{k+1}, 0 < |h - \alpha| \ll 1, h \in (\alpha, \beta),$$

near the center $C(1,0)$. In order to find $b_i(\delta)$, $i = 0,1,2$, we move the center $C_1(1,0)$ into the origin by letting $u = x - 1$, $v = \sqrt{3}y$, ie. $x = \frac{1}{2}u + 1$, $y = \frac{\sqrt{3}}{3}v$, and make the time rescaling $\mathrm{d}\tau = \sqrt{3}\mathrm{d}t$, so that the system (0.1) becomes

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = \frac{\mathrm{d}H_1^c}{\mathrm{d}v} + \varepsilon p_1(u,v), \frac{\mathrm{d}v}{\mathrm{d}\tau} = -\frac{\mathrm{d}H_1^c}{\mathrm{d}u} + \varepsilon q_1(u,v), \tag{2.2}$$

where

$$H_1^c(u,v) = H(x,y) |_{\{x = u+1, y = \frac{\sqrt{3}}{3}v\}} = \frac{1}{4}u^4 + \frac{1}{6}u^2v^2 + \frac{1}{36}v^4 + \frac{2}{3}u^3 + \frac{2}{3}uv^2 + \frac{1}{2}u^2 + \frac{1}{2}v^2 - \frac{1}{12},$$

$$p_1(u,v) = \frac{\sqrt{3}}{3}p(x,y) |_{\{x = u+1, y = \frac{\sqrt{3}}{3}v\}}, q_1(u,v) = \frac{1}{2}q(x,y) |_{\{x = u+1, y = \frac{\sqrt{3}}{3}v\}}.$$

Let

$$\widetilde{M}_1^c(h,\delta) = \oint_{H_1^c(u,v) = h} q_1 \mathrm{d}u - p_1 \mathrm{d}v = \sum_{k \geq 0} b_k^* (h + \frac{1}{12})^{k+1},$$

which is the Melnikov function of the new system (2.2). So we can use the formula in Lemma 1.3 for the Hopf coefficients $b_0^*(\delta)$, $b_1^*(\delta)$, $b_2^*(\delta)$. We obtain

$$b_0^*(\delta) = \frac{4}{3}\pi\sqrt{3}a_0 + \frac{8}{3}\pi\sqrt{3}a_1 + \frac{4}{3}\pi\sqrt{3}a_3 + \frac{16}{3}\pi\sqrt{3}a_4,$$

$$b_1^*(\delta) |_{b_0^* = 0} = -\frac{16}{3}\pi\sqrt{3}a_1 - 4\pi\sqrt{3}a_3 - 16\pi\sqrt{3}a_4,$$

$$b_2^*(\delta) |_{b_0^* = 0} = -\frac{784}{27}\pi\sqrt{3}a_1 - \frac{52}{3}\pi\sqrt{3}a_3 - \frac{208}{3}\pi\sqrt{3}a_4.$$

Note that

$\det \frac{\partial(u,v)}{\partial(x,y)} = \sqrt{3}$ and $k = \sqrt{3}$, we can find $M_1(h,\delta) = \widetilde{M}_1^c(h,\delta)$.

Therefore

$$B_0(\delta) = \frac{4}{3}\pi\sqrt{3}a_0 + \frac{8}{3}\pi\sqrt{3}a_1 + \frac{4}{3}\pi\sqrt{3}a_3 + \frac{16}{3}\pi\sqrt{3}a_4,$$

$$B_1(\delta) |_{B_0 = 0} = -\frac{16}{3}\pi\sqrt{3}a_1 - 4\pi\sqrt{3}a_3 - 16\pi\sqrt{3}a_4,$$

$$B_2(\delta) |_{B_0 = 0} = -\frac{784}{27}\pi\sqrt{3}a_1 - \frac{52}{3}\pi\sqrt{3}a_3 - \frac{208}{3}\pi\sqrt{3}a_4. \tag{2.3}$$

Solve the equation

$$B_0(\delta) = B_1(\delta) = 0,$$

we have

$$a_1 = -\frac{3}{2}a_0, a_4 = \frac{1}{2}a_0 - \frac{1}{4}a_3.$$

We take $\delta = \{a_0, a_1, a_2, a_3, a_4\}$ as well, and $\delta_0 = \{a_0, -\frac{3}{2}a_0, a_2, a_3, \frac{1}{2}a_0 - \frac{1}{4}a_3\}$, by formula(2.1) and

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证法二:

类似证法一可知 $f(x)$ 为单射,且 $f(f(x))=x$, $f(0)=0$. 令 $y=f(-\frac{f(x)}{x})$, $x\neq 0$, 则有 $f(x+f(x)+xf(f(-\frac{f(x)}{x})))=2x+xf(f(-\frac{f(x)}{x}))$. 所以有 $f(x)=x$.

因此,函数方程的解为 $f(x)=x$.

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formula(2.3), it is not difficulty to find $B_2(\delta_0)=\frac{80}{9}\pi\sqrt{3}a_0$, $c_0(\delta_0)=\frac{40}{729}\pi a_0$.

Note that

$$\text{rank}\frac{\partial(B_0,B_1)}{\partial(a_0,a_1,a_2,a_4)}=2,$$

by the lemma 1.3, there are 2 limit cycles near the center C_1, and considering the system is Z_3-equivariant, we have got 6 limit cycyles, bifurcated from the centers.

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