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Central Sets and Radii of the Zero-divisor Graph of Gaussian Integers Modulo n*

模 n 高斯整数环的零因子图的中心集和半径

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Abstract: The central sets and radii of the zero-divisor graph of Gaussian integers modulo n were studied, and the sufficient and necessary conditions were obtained as the radii of zero-divisor graph of Gaussian integers modulo n are 0, 1 and 2, respectively. Meanwhile, the central sets of the zero-divisor graph of Gaussian integers modulo n were found for each positive integer n.

Key words: zero-divisor graph, Gaussian integers modulo n, center, radii

摘要:研究模 n 高斯整数环的零因子图的中心集和半径,得到模 n 高斯整数环的零因子图半径为 0、1、2 时的充要条件,同时对每一个正整数 n,给出模 n 高斯整数环的零因子图的中心集。

关键词:零因子图　模 n 高斯整数环　中心　半径

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All rings considered in this paper will be commutative rings with identity 1. Recall that an element x of a ring R is said to be a zero-divisor if there exists a non-zero element y of R such that $xy=0$. We will use $Z(R)$ to denote the set of zero-divisors of a commutative ring R. The zero-divisor graph of R, denoted by $\Gamma(R)$, is the undirected simple graph with vertices $Z(R)^*=Z(R)-\{0\}$, and for distinct $x, y\in Z(R)^*$, x and y are adjacent if and only if $xy=0$. Note that $\Gamma(R)$ is the empty graph if and only if R is a domain. Moreover, a nonempty $\Gamma(R)$ is finite if and only if R is finite and not a field[1]. The concept of a zero-divisor graph was introduced by Beck I[2]. However, he let all the elements of R be vertices of the graph, and he was mainly interested in colorings. Our definition of $\Gamma(R)$ is from reference [1]. The zero-divisor graph of a commutative ring has also been studied by several other authors[3～7].

The set $Z_n[i]=\{a+bi\mid a,b\in Z_n, i^2=-1\}$ is a commutative ring with the operations modulo n addition and product. The ring $Z_n[i]$ is called the ring of Gaussian integers modulo n. Su et al[3] studied the prime spectrum and zero-divisor of $Z_n[i]$, furthermore, the authors investigated the properties of the zero-divisor graph $\Gamma(Z_n[i])$, including the diameter, the girth, planarity, genus and Eulerian graph[4～6]. In this paper, we continue to explore the properties of the zero-divisor graph $\Gamma(Z_n[i])$, to be more precise, the central sets and the radii are found for each positive integer n.

For convenience, we introduce some basic nations. For any vertex x of a connected simple graph G, the eccentricity of x, denoted by $\varepsilon(x)$, is the maximum of the distances from x to the other vertices of G. The set of vertices with minimal eccentricity is called the center of the graph, denoted by G_c, and

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this minimum eccentricity value is the radius of G, denoted by $r(G)$. The diameter of a connected graph G, denoted by diam (G), is the maximum value of $\varepsilon(x)$ for every vertices of G. It is well known that if a connected graph G has radius $r(G)$ and diameter diam (G), then $r(G) \leqslant \text{diam}(G) \leqslant 2r(G)$. For the other graphic nations, please refer to reference[8].

1 The radii of $\Gamma(Z_n[i])$

It has been established that the radius of the zero-divisor graph of a commutative ring is either 0, 1, or 2[7]. If the radius of the zero-divisor graph is 0, then the graph consists of a single vertex. The radius of the zero-divisor graph is 1 if and only if there exists a vertex linked all other vertices in the graph. According to this, we will study when the radii are 0, 1, and 2, respectively in $\Gamma(Z_n[i])$ in this section. Firstly, we give some special cases.

Theorem 1.1 Let $R = Z_{2^n}[i], n \geqslant 2$. Then the radius of $\Gamma(R)$ is 1.

Proof If $n \geqslant 2$, then $Z(R) = \{(1+i)\alpha \mid \alpha \in R\}$ by the Theorem 3.1(1) of reference[3]. For any $(1+i)\alpha \in Z(R)^*$, since $2^{n-1}(1+i)(1+i)\alpha = 2^n i a = 0$, we have $\varepsilon(2^{n-1}(1+i)) = 1$. So the radius of $\Gamma(R)$ is 1.

Theorem 1.2 Let $R = Z_{p^n}[i]$, where $p \equiv 3(\text{mod } 4)$ is a prime and $n \geqslant 2$, then the radius of $\Gamma(R)$ is 1.

Proof From the Theorem 3.1(2) of reference [3], we know that R is a local ring with the unique maximal ideal $m = \langle p \rangle$. So $Z(R) = m = \{p\alpha \mid \alpha \in R\}$. For any $p\alpha \in Z(R)^*$, since $p^{n-1} p\alpha = p^n \alpha = 0$, we have $\varepsilon(p^{n-1}) = 1$. So the radius of $\Gamma(R)$ is 1.

Theorem 1.3 Let $R = Z_{p^n}[i]$, where $p \equiv 1(\text{mod } 4)$ is a prime and $n \geqslant 1$, then the radius of $\Gamma(R)$ is 2.

Proof According to the Theorem 3.1(3) of reference[3], R is a semi-local ring with two maximal ideals, $m_1 = \langle a+bi \rangle$ and $m_2 = \langle a-bi \rangle$, where $p = (a+bi)(a-bi)$. If $n = 1$, then $\Gamma(R)$ is a bipartite graph, so the radius of $\Gamma(R)$ is 2. If $n \geqslant 2$, then diam $\Gamma(R) = 3$, by the Theorem 1 of reference [4], so the radius of $\Gamma(R)$ is also 2. This completes our proof.

Theorem 1.4 Let $R = Z_n[i], n \geqslant 2$. Suppose that $\Gamma(R)$ is not an empty graph. Then

(1) The radius of $\Gamma(R)$ is 0 if and only if $n = 2$.

(2) The radius of $\Gamma(R)$ is 1 if and only if $n = 2^k$ or $n = p^k$, where $p \equiv 3(\text{mod } 4)$ is a prime and $k \geqslant 2$.

(3) The radius of $\Gamma(R)$ is 2 if and only if $n \neq 2^k$ for any k, and $n \neq p^k$ where $p \equiv 3(\text{mod}4)$ is a prime.

Proof When $n = 2$, then the zero-divisors of $Z_n[i]$ are 0 and $1+i$, so $\Gamma(R)$ just contains a vertex, thus the radius of $\Gamma(R)$ is 0. Conversely, if the radius of $\Gamma(R)$ is 0, then $\Gamma(R)$ contains only one vertex, so $n = 2$.

By the Theorem 1.2 and 1.3, we know that radius of $\Gamma(R)$ is 1 if $n = 2^k$ or $n = p^k$, where $p \equiv 3 (\text{mod}4)$ is a prime and $k \geqslant 2$. Now Assume that radius of $\Gamma(R)$ is 1, then there is one element in $Z(R)^*$, such that it adjacent to every other vertex. Then $R \cong Z_2 \times F$, where F is a finite field or R is a local ring by the Theorem 2.1 of reference[2]. But the case $R \cong Z_2 \times F$ does not occur since $R = Z_n[i]$, so R is local, consequently, $n = 2^k$ or $n = p^k$, where $p \equiv 3 (\text{mod } 4)$ is a prime and $k \geqslant 2$. By the Theorem 2.3 of reference[7], the radius of $\Gamma(R)$ is at most 2, so this completes our proof.

2 The center of $\Gamma(Z_n[i])$

If the radius of the zero-divisor graph is 1, then those elements in the center are precisely those elements of eccentricity 1. Firstly we also study some special cases.

Theorem 2.1 Let $R = Z_{2^n}[i], n \geqslant 1$, and $G = \Gamma(R)$. Then $G_c = \{2^{n-1}(1+i)\}$.

Proof If $n = 1$, then $\Gamma(R)$ is a singleton graph, so the statement holds. If $n \geqslant 2$, then, by the Theorem 3.1(1) of reference[3], $Z(R) = \{(1+i)\alpha \mid \alpha \in R\}$. For all $(1+i)\alpha \in Z(R)^*$, since $2^{n-1}(1+i)(1+i)\alpha = 2^n i a = 0$, we have $\varepsilon(2^{n-1}(1+i)) = 1$.

On the other hand, let $u = (1+i)\alpha \in Z(R)^*$ and $\varepsilon(u) = 1$, then we have $(1+i)\alpha \cdot (1+i)\beta = 0$, for all $(1+i)\beta \in Z(R)^*$, i.e., $2i\alpha\beta = 0$, by the randomicity of β, $2^{n-1} \mid \alpha$. So $u = 2^{n-1}(1+i)$. This completes the proof.

Theorem 2.2 Let $R = Z_{p^k}[i]$, where $p \equiv 3(\text{mod } 4)$ is a prime, and $k \geqslant 2$. Suppose $G = \Gamma(R)$ and $I = \langle p^{k-1} \rangle$. Then $G_c = I^*$.

Proof If $k = 2$, then $\Gamma(R)$ is a complete graph, so the statement holds. If $k \geqslant 3$, then $Z(R) = \{p\alpha \mid \alpha \in R\}$ by the Theorem 3.1(2) of reference [3]. From the theorem 1 of reference [4], we know that diam $(\Gamma(R)) = 2$. So $\varepsilon(u) \leqslant 2$ for all $u \in \Gamma(R)$. Let $\varepsilon(u) = 1$, then for all $p\alpha \in Z(R)^*$, $u \cdot p\alpha = 0$, by the randomicity of α, $p^{k-1} \mid u$. So $u \in \langle p^{k-1} \rangle$.

On the other hand, for all $\nu \in \langle p^{k-1} \rangle$, we have $\nu \cdot p\alpha = 0$ for all $p\alpha \in Z(R)^*$, so $\varepsilon(\nu) = 1$. Hence the Theorem 2.2 holds.

Theorem 2.3 Let $R = Z_{p^k}[i]$, where $p \equiv 1 \pmod 4$ is a prime, and $k \geqslant 1$, $G = \Gamma(R)$. Then $G_c = Z(R)^*$ if $k = 1$ and $G_c = J^* = \langle p \rangle - \{0\}$ if $k \geqslant 2$.

Proof Assume $n = p^k$, $p \equiv 1 \pmod 4$. If $k = 1$ then $\Gamma(R)$ is a complete bipartite graph, so the statement holds. If $k \geqslant 2$, then by the Theorem 1 of reference [4], diam $(\Gamma(R)) = 3$, so for all $u \in \Gamma(R)$, $2 \leqslant \varepsilon(u) \leqslant 3$. By the Theorem 3.1(3) of reference [3], $Z(R) = m_1 \cup m_2$, where $m_1 = \langle a + bi \rangle$, $m_2 = \langle a - bi \rangle$ are two maximal ideals of R and $a^2 + b^2 = p$. If $u \in \langle p \rangle$, let $u = p\alpha$, then for all $\nu \in \Gamma(R)$, if $\nu \in m_1^*$, we have $u \leftrightarrow \omega \leftrightarrow \nu$ is a path with length 2, where $w = p^{k-1}(a - bi)$. If $\nu \in m_2^*$, we have $u \leftrightarrow \omega \leftrightarrow \nu$ is a path with length 2, where $w = p^{k-1}(a + bi)$. So $\varepsilon(u) = 2$. If $u \notin \langle p \rangle$, we may suppose $u = (a + bi)^e$. Obviously, $(a + bi)^e \cdot (a + bi) = (a + bi)^{e+1} \neq 0$, and $N(a - bi) = \{p^{k-1}(a + bi)\alpha \mid \alpha \in R\}$, where $N(a - bi)$ is the set of elements adjacent to $a - bi$. This completes the proof.

Lemma 2.1[7] Let n and m be positive integers, $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$, where each R_i is a commutative Artinian local ring with identity that is not a field and each F_i is a field. For each $j = 1, \cdots, m$, define the ideal $I_j = \{0\} \times \cdots \times \{0\} \times F_j \times \{0\} \times \cdots \times \{0\}$. Then the center of $\Gamma(R)$ is $J(R) \cup (\bigcup_{j=1}^{m} I_j) - \{(0, \cdots, 0)\}$, where $J(R)$ is the Jacobson radical of R.

Theorem 2.4 Let $R = Z_n[i]$, $n = 2^{k_0} p_1^{k_1} \cdots p_s^{k_s} q_1^{l_1} \cdots q_r^{l_r} q_{r+1} \cdots q_{r+t}$, where p_i, q_j are distinct odd prime numbers, and $p_i \equiv 1 \pmod 4$, $q_j \equiv 3 \pmod 4$, $i = 1, \cdots, s$, $j = 1, \cdots, r, \cdots, r + t$, $k_i \geqslant 1$, $i = 1, \cdots, s$, $l_j \geqslant 2$, $j = 1, \cdots, r$, $k_0 \geqslant 0$. Then $\Gamma(R)_c = J \cup (\bigcup_{j=1}^{t} \langle v_{r+j} \rangle) - \{0\}$, where $v_{r+j} = \frac{n}{q_{r+j}}$, $j = 1, \cdots, t$, J is the Jacobson radical of R.

Proof We know that if $k_0 = 0$, then $J = \langle p_1 \cdots p_s q_1 \cdots q_{r+t} \rangle$, and if $k_0 \geqslant 1$, then $J = \langle (1 + i) p_1 \cdots p_s q_1 \cdots q_{r+t} \rangle$ by the Theorem 3.3(3) of reference [3]. Let $n_1 = 2^{k_0} p_1^{k_1} \cdots p_s^{k_s} q_1^{l_1} \cdots q_r^{l_r}$. Then $Z_n[i] \cong Z_{n_1}[i] \oplus Z_{q_{r+1}}[i] \oplus \cdots \oplus Z_{q_{r+t}}[i]$ by the Theorem 3.2 of reference [3]. Suppose $\phi: Z_n[i] \to Z_{n_1}[i] \oplus Z_{q_{r+1}}[i] \oplus \cdots \oplus Z_{q_{r+t}}[i]$ is an isomorphism of rings and let $\phi(x) = (0, \cdots, 0, \alpha_{r+i}, 0, \cdots, 0)$, $\alpha_{r+i} \neq 0$, then $n_1 \mid x$, $q_1 \mid x, \cdots, q_{r+i-1} \mid x$, $q_{r+i+1} \mid x, \cdots, q_{r+t} \mid x$, and $q_{r+i} \nmid x$. This implies $x \in \langle v_{r+i} \rangle$, then by the Lemma 2.1, it completes the proof.

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