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Convergence of the Non-negative Solutions of a Higher-order Rational Difference Equation*

一类高阶有理差分方程非负解的收敛性

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Abstract: This paper is concerned with the following higher-order rational difference equation:

$$x_{n+1}=\alpha+\frac{\sum_{i=1}^{k+1}B_{2i-1}x_{n-2i+1}}{A+x_{n-2l}},n=0,1,\cdots$$

where k and l are non-negative integers, the parameter α is positive real number, the parameters $A,B_i,i=1,2,\cdots,k+1$ and the initial conditions are non-negative real numbers. We give the sufficient conditions, under which every non-negative solution of the equation converges to a period-two solution of the equation.

Key words: difference equation, convergence, period-two solution, boundedness

摘要:针对高阶有理差分方程 $x_{n+1}=\alpha+\frac{\sum_{i=1}^{k+1}B_{2i-1}x_{n-2i+1}}{A+x_{n-2l}},n=0,1,\cdots$,其中 k,l 为非负整数,α 是正数,$A,B_i,i=1,2,\cdots,k+1$ 和初始条件是非负数,给出该方程的每个非负解都收敛于方程的一个二周期解的一个充分条件.

关键词:差分方程　收敛性　二周期解　有界性

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Recently there has been a great interest in studying nonlinear difference equations for developing some techniques, which can be used in investigating the equations arising in models describing real life situations in biology, control theory, economics, etc[1~6]. In reference[1], the following difference equation was investigated:

$$x_{n+1}=1+\frac{x_{n-2k+1}}{x_{n-2l}},n=0,1,\cdots \qquad (0.1)$$

The authors showed that every positive solution of Equation(0.1) converges to a period-two solution of Equation(0.1).

Motivated by the above studies, in this paper, we investigate the following difference equation

$$x_{n+1}=\alpha+\frac{\sum_{i=1}^{k+1}B_{2i-1}x_{n-2i+1}}{A+x_{n-2l}},n=0,1,\cdots \qquad (0.2)$$

where k,l are non-negative integers, the parameter α is a positive real number, the parameters $A,B_i,i=1,2,\cdots,k+1$ and the initial conditions are non-negative real numbers.

Set $X=\{i\mid B_{2i-1}\neq 0,i=1,2,\cdots,k+1\}=\{i_1,i_2,\cdots,i_r\mid i_t<i_m \text{ for } t<m\}$. $q=\sum_{i=1}^{k+1}B_{2i-1},t=$

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$\max\{2k+1,2l\}$, $s=[\frac{t}{2}]$. We assume that the following statement is true:

(H) $r>1$ and $i_1,i_2,\cdots,i_r$ are relatively prime.

Under this condition, we show that every solution of Equation(0.2) converges to a period-two solution of Equation(0.2) if and only if

$$q=A+\alpha.$$

Since $i_1,i_2,\cdots,i_r$ are relatively prime, there exist non-zero integers $\lambda_1,\lambda_2,\cdots,\lambda_r$ such that

$$\lambda_1 i_1+\lambda_2 i_2+\cdots+\lambda_r i_r=1.$$

In the sequel, for convenience, set

$N_0=s(|\lambda_1|i_1+|\lambda_2|i_2+\cdots+|\lambda_r|i_r)$, then for $-s\leqslant j\leqslant s$,

$$N_0+j=s(|\lambda_1|i_1+|\lambda_2|i_2+\cdots+|\lambda_r|i_r)+j=(s|\lambda_1|+j\lambda_1)i_1+(s|\lambda_2|+j\lambda_2)i_2+\cdots+(s|\lambda_r|+j\lambda_r)i_r.$$

1 Preliminaries

Now some definitions and known results are presented which will be useful in the investigation.

Let $f:J^{k+1}\to J$ be a continuous function, where k is a non-negative integer and J is an interval of real numbers. Consider the difference equation

$$x_{n+1}=f(x_n,x_{n-1},\cdots,x_{n-k}),n=0,1,\cdots \quad (1.1)$$

with initial conditions $x_{-k},x_{-k+1},\cdots,x_0\in J$.

We say that $\{x_n\}_{n=-k}^{\infty}$ is a period-two solution of Equation(1.1) if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Equation (1.1) and

$$x_n=x_{n+2} \text{ for all } n\geqslant -k.$$

The following result about Full Limiting Sequence is due to Karakostas.

Theorem 1.1[2] Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Equation(1.1). Set $I=\liminf_{n\to\infty}x_n$, $S=\limsup_{n\to\infty}x_n$ and suppose that $I,S\in J$. Let L be a limit point of the sequence $\{x_n\}_{n=-k}^{\infty}$. Then the following statements are true:

(1) There exists a solution $\{L_n\}_{n=-\infty}^{\infty}$ of Equation (1.1), called a full limiting sequence of $\{x_n\}_{n=-k}^{\infty}$, such that $L_0=L$, and such that for every $m\in\mathbb{Z}$, L_m is a limit point of $\{x_n\}_{n=-k}^{\infty}$.

(2) For every $i_0\in\mathbb{Z}$, there exists a subsequence $\{x_{r_i}\}_{i=0}^{\infty}$ of $\{x_n\}_{n=-k}^{\infty}$ such that $L_n=\lim_{i\to\infty}x_{r_i+n}$ for every $n\geqslant i_0$.

2 Main results

Lemma 2.1 The following statements are true:

(1) There exists a non-negative solution of Equation(0.2), which is periodic with prime period-two if and only if

$$q=A+\alpha.$$

(2) Suppose $q=A+\alpha$. Then

$$\phi,\psi,\phi,\psi,\cdots$$

is a solution of Equation(0.2), which is periodic with prime period-two if and only if

$$\alpha(A+\phi+\psi)=\phi\psi \text{ and } \phi\neq\psi.$$

The proof of Lemma 2.1 follows by simple computations and will be omitted.

Lemma 2.2 Suppose $q=A+\alpha$. Then every non-negative solution of Equation(0.2) is bounded.

Proof Let $\{x_n\}_{n=-t}^{\infty}$ be a non-negative solution of Equation(0.2). Choose $m,M>0$ such that

$$m\leqslant\min\{x_1,x_2,\cdots,x_{t+1}\}\leqslant\max\{x_1,x_2,\cdots,x_{t+1}\}\leqslant M,\alpha(A+M+m)=Mm.$$

From Equation(0.2) we get

$$m=\alpha+\frac{qm}{A+M}\leqslant\alpha+\frac{\sum_{i=1}^{k+1}B_{2i-1}x_{t-2i+2}}{A+x_{t-2l+1}}\leqslant\alpha+\frac{qM}{A+m}=M,$$

i.e.

$$m\leqslant x_{t+2}\leqslant M.$$

It follows by induction that

$$m\leqslant x_n\leqslant M \text{ for all } n\geqslant 1.$$

Hence $\{x_n\}_{n=-t}^{\infty}$ is bounded, and the proof is complete.

Remark Let $\{x_n\}_{n=-t}^{\infty}$ be a non-negative solution of Equation(0.2). It follows by Lemma 2.2 that $0<\liminf_{n\to\infty}x_n\leqslant\limsup_{n\to\infty}x_n<\infty$. Throughout the remainder of this section, set

$$I=\liminf_{n\to\infty}x_n \text{ and } S=\limsup_{n\to\infty}x_n.$$

Theorem 2.1 Every solution of Equation(0.2) converges to a period-two solution of Equation (0.2).

Proof Let $\{x_n\}_{n=-t}^{\infty}$ be a non-negative solution of Equation(0.2). Note that $q=A+\alpha$, hence

$$S\leqslant\alpha+\frac{(A+\alpha)S}{A+I},I\geqslant\alpha+\frac{(A+\alpha)I}{A+S}.$$

It flows that

$\alpha(A+S+I)=SI$.

Set $m_0=2s\max\{|\lambda_1|,|\lambda_2|,\cdots,|\lambda_r|\}$. Suppose, without loss of generality that there is a subsequence $\{x_{2k_j}\}_{j=0}^{\infty}$ of $\{x_n\}_{n=-t}^{\infty}$ such that

$$\lim_{j\to\infty}x_{2k_j}=I_0=I,\lim_{j\to\infty}x_{2k_j-2}=I_1,\lim_{j\to\infty}x_{2k_j-4}=I_2,\cdots,\lim_{j\to\infty}x_{2k_j-2(m_0+l+1+N_0+k)}=I_{m_0+l+1+N_0+k},\qquad(2.1)$$

$$\lim_{j\to\infty}x_{2k_j-1}=S_1,\lim_{j\to\infty}x_{2k_j-3}=S_2,\cdots,\lim_{j\to\infty}x_{2k_j-2(m_0+l+1+N_0+k)+1}=S_{m_0+l+1+N_0+k}.\qquad(2.2)$$

Note that

$$x_{2k_j}=\alpha+\frac{\sum_{i=1}^{k+1}B_{2i-1}x_{2k_j-2i}}{A+x_{2k_j-2l-1}},$$

by taking limits, as $j\to\infty$, we have

$$I=\alpha+\frac{\sum_{i=1}^{k+1}B_{2i-1}I_i}{A+S_{l+1}}\geqslant\alpha+\frac{(A+\alpha)I}{A+S}=I.$$

Note that $I\leqslant I_i,S_i\leqslant S$ for all $i\in\{1,2,\cdots,m_0+l+1+N_0+k\}$, we have

$$\sum_{i=1}^{k+1}B_{2i-1}I_i=(A+\alpha)I,S_{l+1}=S.$$

Since

$$B_{2i_j-1}\neq 0 \text{ for } j=1,2,\cdots,r,$$

we have

$$I_{i_j}=I \text{ for } j=1,2,\cdots,r.$$

It follows by induction that for $p_1,p_2,\cdots,p_r\in\{0,1,2,\cdots\}$,

$$I_{p_1i_1+p_2i_2+\cdots+p_ri_r}=I,S_{l+1+p_1i_1+p_2i_2+\cdots+p_ri_r}=S.$$

For $-s\leqslant j\leqslant s$, note that

$$N_0+j=(s|\lambda_1|+j\lambda_1)i_1+(s|\lambda_2|+j\lambda_2)i_2+\cdots+(s|\lambda_r|+j\lambda_r)i_r,$$

we have

$$I_{N_0+j}=I,S_{l+1+N_0+j}=S \text{ for } -s\leqslant j\leqslant s.\qquad(2.3)$$

Similarly, by $S_{l+1}=S$, we get $I_{2l+1}=I$, from which we get

$$I_{2l+1+N_0+j}=I \text{ for } -s\leqslant j\leqslant s.\qquad(2.4)$$

Now we claim that there exists $c\in N$ such that

$$I_i=I,S_i=S \text{ for } c-s\leqslant i\leqslant c.\qquad(2.5)$$

In fact, Case 1: $l<k$. Then $s=k$, take $c=N_0+s$, note that formula(2.3), hence formula(2.5) holds. Case 2: $l=k$. Then $s=k=l$, take $c=N_0+2s+1$, note that formula(2.3) and(2.4), hence formula (2.5) holds. Case 3: $l>k$. Then $s=l$, take $c=N_0+2l+1$, note that formula(2.3) and (2.4), hence formula(2.5) holds.

Let $\epsilon>0$ be given, set $\epsilon_1=\frac{(S-\alpha)\epsilon}{I+\epsilon-\alpha}<\frac{S-\alpha}{I-\alpha}\epsilon$.

Then, in view of formula(2.1) and (2.2), there exists $i\geqslant 0$, such that

$$x_{2k_i-2c},x_{2k_i-2(c-1)},\cdots,x_{2k_i-2(c-s)}<I+\epsilon,$$

and

$$x_{2k_i-2c+1},x_{2k_i-2(c-1)+1},\cdots,x_{2k_i-2(c-s)+1}>S-\epsilon_1.$$

Hence

$$x_{2k_i-2(c-s)+2}<\alpha+\frac{q(I+\epsilon)}{A+S-\epsilon_1}=I+\epsilon,$$

$$x_{2k_i-2(c-s)+3}>\alpha+\frac{q(S-\epsilon_1)}{A+I+\epsilon}=S-\epsilon_1.$$

It follows by induction that for all $n\geqslant 2k_i-2c$, we have

$x_n<I+\epsilon$, when n is even;

$x_n>S-\epsilon_1$, when n is odd.

And so clearly

$$\lim_{n\to\infty}x_{2n}=I,\lim_{n\to\infty}x_{2n+1}=S.$$

The proof is complete.

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