◆特邀专稿◆

Cycles Embedding in d-Ary n-Dimensional Cube With Node Failures*

LI Zhaoxiang
(College of Science, Minzu University of China, Beijing, 100081, China)

Abstract: The d-ary n-dimensional cube (the general form of hypercube) has been widely used as the interconnection network in parallel computers. The fault-tolerant capacity of an interconnection network is a critical issue in parallel computing. In this article, we consider the fault-tolerant capacity of the d-ary n-dimensional cube. Let F be a set of faulty vertices in $Q_n(d)$ $(n \geqslant 3)$ with $|F| \leqslant n-2$, we prove that every fault-free edge and fault-free vertex (node) of $Q_n(d)$ lies on a fault-free cycle of every even length from 4 to $d^n - 2|F|$. Moreover, if d is an odd number, every fault-free edge and fault-free vertex (node) of $Q_n(d)$ lies on a fault-free cycle of length $d^n - 2|F|$.

Key words: cycle embedding, hypercube, fault-tolerant, interconnection network, d-ary

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0 Introduction

Network topology is usually represented by a graph where vertices represent processor and edges represent links between processors[1]. The hypercube has been widely used as the interconnection network in parallel computers[2,3]. The n-dimensional generalized hypercube, denoted by $Q(d_1, d_2, \cdots, d_n)$, where $d_i (\geqslant 2)$ is an integer for each $i = 1, 2, \cdots, n$. The vertex-set of $Q(d_1, d_2, \cdots, d_n)$ is the set $V = \{x_1 x_2 \cdots x_n : x_i \in \{0, 1, \cdots, d_i - 1\}, i = 1, 2, \cdots, n\}$ and two vertices $x = x_1 x_2 \cdots x_n$ and $y = y_1 y_2 \cdots y_n$ are linked by an edge if and only if they differ exactly in one coordinate. If $d_1 = d_2 = \cdots = d_n = d \geqslant 2$, then $Q(d, d, \cdots, d)$ is called the d-ary n-dimensional cube, denoted by $Q_n(d)$. It is clear that $Q_n(2)$ is hypercube Q_n. For two vertices u and v in $Q_n(d)$, the Hamming distance $h(u, v)$ between two vertices u and v is the number of different bits in the corresponding strings of both vertices; and

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【作者简介】
李赵祥(1968-),男,博士,教授,主要从事图论及应用研究,E-mail:zhaoxiangli8@163.com。

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the distance between u and v, denoted by $D(Q_n(d);u,v)$, is the length of the shortest path between u and v. Obviously, $h(u,v)=D(Q_n(d);u,v)$. Let $u=u_1u_2\cdots u_n$ be a vertex of $Q_n(d)$, $u^{j(a)}=v=v_1v_2\cdots v_n$ is also a vertex of $Q_n(d)$, $v_i=u_i(1\leqslant i\leqslant n, i\neq j, j\in\{1,2,\cdots,n\})$, $v_j\neq u_j$, $v_j=a\in\{0,1,2,\cdots,d-1\}$. A vertex is fault-free if it is not faulty. An edge is fault-free if the two end-vertices and the link between them are not faulty. A cycle of length k is called k-cycle. A graph G is vertex-transitive if for any given pair (x,y) of vertices in G there is some $\theta\in \mathrm{Aut}(G)$ ($\mathrm{Aut}(G)$ is an automorphism group of G) such that $y=\theta(x)$.

The cycle embedding problem deals with all possible lengths of the cycles in a given graph, it is investigated in a lot of interconnection networks[4]. The fault-tolerant capacity of an interconnection network is a critical issue in parallel computing[2]. For hypercube Q_n, Saad and Schultz[5] proved that an even cycle of length k exists for each even integer between 4 and 2^n. Let f_e (respectively, f_v) be the number of faulty edges (respectively, vertices) in Q_n. If $f_e\leqslant n-2$, Li et al.[1] proved that every fault-free edge of $Q_n(n\geqslant 3)$ lies on a fault-free cycle of every even length from 4 to 2^n. If $f_e\leqslant n-1$ and all faulty edges are not incident with the same vertex, Xu et al.[6] showed that every fault-free edge of $Q_n(n\geqslant 4)$ lies on a fault-free cycle of every even length from 6 to 2^n. Fu[7] proved that a fault-free cycle of length with at least 2^n-2f_v can be embedded in Q_n with $f_v\leqslant 2n-4$. If $f_v\leqslant 2n-2$, Tsai[2] proved that every fault-free edge and fault-free vertex of Q_n lies on a fault-free cycle of every even length from 4 to 2^n-2f_v. Stewart and Xiang[8] studied the bipanconnectivity and bipancyclicity in k-ary n-cubes. Cheng et al.[9] studied the vertex-fault-tolerant cycles embedding in balanced hypercubes with faulty edges; Hao et al.[10] studied the hamiltonian cycle embedding for fault tolerance in balanced hypercubes.

In this article, we study the cycle embedding in $Q_n(d)$. For any subset F of $V(Q_n(d))(n\geqslant 3)$ with $|F|\leqslant n-2$, we prove that every fault-free edge and fault-free vertex (node) of $Q_n(d)$ lies on a fault-free cycle of every even length from 4 to $d^n-2|F|$. If $d=2$, these results are the results of Tsai[2].

1 Preliminaries

The *n-bit* Gray code is a ring sequence of *n-bit* numbers (the number of each coordinate is selected from $\{0,1,2,\cdots,d-1\}$) such that any two successive numbers have one and only one different bit and so that all numbers having n bits are represented. The *n-bit* Gray code is denoted by G_n. If d is an even number. One starts with the sequence of the d 1-*bit* numbers $0,1,2,\cdots,d-1$. This is a 1-*bit* Gray code, *i.e.*, $G_1=\{0,1,2,\cdots,d-1\}$. To obtain a 2-*bit* Gray code G_2, take the same sequence and insert a zero in front of each number, then take the sequence in reverse order and insert a one in front of each number, take the same sequence and insert a 2 in front of each number, then take the sequence in reverse order and insert a 3 in front of each number, take the same sequence and insert a $d-2$ in front of each number, then take the sequence in reverse order and insert a $d-1$ in front of each number. In other words, from $G_1=\{0,1,2,\cdots,d-1\}$, we get a 2-*bit* Gray code $G_2=\{00,01,\cdots,0(d-2),0(d-1),1(d-1),1(d-2),\cdots,11,10,\cdots,(d-2)0,(d-2)1,\cdots,(d-2)(d-2),(d-2)(d-1),(d-1)(d-1),(d-1)(d-2),\cdots,(d-1)1,(d-1)0\}$. More generally, denoted by G_n^R the sequence obtained from G_n by reversing its order, and by mG_n, $m=0,1,2,\cdots,d-1$ (respectively, mG_n^R) the sequence obtained from G_n by inserting a m in front of each element of the sequence, then an $(n+1)-bit$ Gray code can be generated by the recursion $G_{n+1}=\{0G_n,1G_n^R,2G_n,3G_n^R,\cdots,(d-2)G_n,(d-1)G_n^R\}$. If d is an odd number, Gray codes can be similar to generate.

Let V_n be the set of vertices of $Q_n(d)$. For a given $i(0\leqslant i\leqslant d-1)$, let iV_{n-1} be the subset of vertices of $Q_n(d)$ whose fist coordinate is i. Thus the set of vertices of $Q_n(d)$ can be decomposed into d disjoint subsets $0V_{n-1},1V_{n-1},\cdots,(d-1)V_{n-1}$. We use $iQ_{n-1}(d)$ to denote the subgraph of $Q_n(d)$ induced by iV_{n-1}. Then $iQ_{n-1}(d)$ is isomorphic to

$Q_{n-1}(d)$. It is often convenient to write $Q_n(d) = 0Q_{n-1}(d)\Theta 1Q_{n-1}(d)\Theta\cdots\Theta(d-1)Q_{n-1}(d)$.

Lemma 1 Let u and v be two distinct vertices of $Q_n(d)$. Then, there is a partition which can partition $Q_n(d)$ into d copies $Q_{n-1}(d)$, denoted by $Q^i_{n-1}(d)(i\|0,1,\cdots,d-1)$ such that $u\in V(Q^m_{n-1}(d))$ and $v\in V(Q^k_{n-1}(d))(m,k\in\{0,1,2,\cdots,d-1\}, m\neq k\}$.

Proof Let $u=u_1u_2\cdots u_n$ and $v=v_1v_2\cdots v_n$. Since u and v are distinct vertices, there is an index $j(j\in\{1,2,\cdots,n\}$ such that $u_j\neq v_j$, $u_j\in\{0,1,\cdots,d-1\}$, $v_j\in\{0,1,\cdots,d-1\}$. Therefore, $Q_n(d)$ can be partitioned along dimension j into d copies $Q_{n-1}(d)$ such that one contains u and the other contains v.

Lemma 2 Let $e=(u,v)$ be an edge of $Q_n(d)$. Then, there is a partition which can partition $Q_n(d)$ into d copies $Q_{n-1}(d)$, denoted by $Q^i_{n-1}(d)$ $(i=0,1,\cdots,d-1)$ such that $u\in V(Q^m_{n-1}(d))$ and $v\in V(Q^m_{n-1}(d))(m\in\{0,1,2,\cdots d-1\})$, i.e., e is an edge of $Q^m_{n-1}(d)$.

Proof Let $e=(u,v)$ be an edge of $Q_n(d)$, $u=u_1u_2\cdots u_n$, $v=v_1v_2\cdots v_n$, then, there is an index i $(i\in\{1,2,\cdots,n\})$ such that $u_i\neq v_i$, $u_j=v_j(1\leqslant j\leqslant n, j\neq i)$. Therefore, $Q_n(d)$ can be partitioned along dimension j into d copies $Q_{n-1}(d)$ such that $e\in E(Q^m_{n-1}(d))(m\in\{0,1,2,\cdots,d-1\})$.

2 d is an even number

Theorem 1 Let x and y be any two vertices in $Q_n(d)(n\geqslant 2)$ and l be any integer with $D(Q_n(d);x,y)\leqslant l\leqslant d^n-1$. If d is an even number and $l-D(Q_n(d);x,y)$ is also an even number, then there is an xy-path of length l in $Q_n(d)$.

Proof Let $D(Q_n(d);x,y)=m$. The proof is based on the recursive structure of $Q_n(d)$ by induction on $n\geqslant 2$. When $n=2$, if $D(Q_2(d);x,y)=1$. By the vertex-transitivity of $Q_2(d)$[3], without loss of generality, we can assume $x=00$, $y=01$.

$x=00\to 01=y$, $x=00\to 02\to 03\to 01=y$, $x=00\to 02\to 03\to 04\to 05\to 01=y$, $\cdots$, $x=00\to 02\to 03\to 04\to 05\to\cdots\to 0(d-2)\to 0(d-1)\to 01=y$ are the xy-path of length $l=1,3,5,\cdots,d-1$ in $Q_2(d)$.

$x=00\to 10\to 12\to 02\to 03\to 04\to 05\to\cdots\to 0(d-2)\to 0(d-1)\to 01=y$. $x=00\to 10\to 20\to 22\to 12\to 02\to 03\to 04\to 05\to\cdots\to 0(d-2)\to 0(d-1)\to 01=y$. $\cdots$. $x=00\to 10\to 20\to 30\to 40\to\cdots\to(d-2)0\to(d-1)0\to(d-1)2\to(d-2)2\to\cdots\to 22\to 12\to 02\to 03\to 04\to 05\to\cdots\to 0(d-2)\to 0(d-1)\to 01=y$. $\cdots$. $x=00\to 10\to 20\to 30\to 40\to\cdots\to(d-2)0\to(d-1)0\to(d-1)2\to(d-2)2\to\cdots\to 22\to 12\to 02\to 03\to 13\to 23\to\cdots\to(d-2)3\to(d-1)3\to(d-1)4\to(d-2)4\to\cdots\to 24\to 14\to 04\to 05\to\cdots\to 0(d-1)\to 1(d-1)\to 2(d-1)\to\cdots\to(d-2)(d-1)\to(d-1)(d-1)\to(d-1)1\to(d-2)1\to\cdots\to 21\to 11\to 01=y$ are the xy-path of length $l=d+1,d+3,\cdots,3(d-1),\cdots,d^2-1$ in $Q_2(d)$.

When $n=2$, if $D(Q_2(d);x,y)=2$. By the vertex-transitivity of $Q_2(d)$[3], without loss of generality, we can assume $x=00$, $y=11$.

$x=00\to 10\to 11=y$, $x=00\to 20\to 30\to 10\to 11=y$. $x=00\to 20\to 30\to 40\to 50\to 10\to 11=y$. $\cdots$. $x=00\to 20\to 30\to 40\to 50\to\cdots\to(d-2)0\to(d-1)0\to 10\to 11=y$ are the xy-path of length $l=2,4,6,\cdots,d$ in $Q_2(d)$.

$x=00\to 01\to 21\to 20\to 30\to 40\to 50\to\cdots\to(d-2)0\to(d-1)0\to 10\to 11=y$. $x=00\to 01\to 02\to 22\to 21\to 20\to 30\to 40\to 50\to\cdots\to(d-2)0\to(d-1)0\to 10\to 11=y$. $\cdots$. $x=00\to 01\to 02\to\cdots\to 0(d-2)\to 0(d-1)\to 2(d-1)\to 2(d-2)\to\cdots\to 22\to 21\to 20\to 30\to 40\to\cdots(d-3)0\to(d-2)0\to(d-1)0\to 10\to 11=y$. $\cdots$. $x=00\to 01\to 02\to\cdots\to 0(d-2)\to 0(d-1)\to 2(d-1)\to 2(d-2)\to\cdots\to 22\to 21\to 20\to 30\to 31\to 32\to\cdots\to 3(d-2)\to 3(d-1)\to 4(d-1)\to 4(d-2)\to\cdots\to 42\to 41\to 40\to\cdots\to(d-3)0\to(d-3)1\to(d-3)2\to\cdots\to(d-3)(d-2)\to(d-3)(d-1)\to(d-2)(d-1)\to(d-2)(d-2)\to\cdots\to(d-2)2\to(d-2)1\to(d-2)0\to(d-1)0\to(d-1)2\to(d-1)3\to\cdots\to(d-1)(d-2)\to(d-1)(d-1)\to 1(d-1)\to 1(d-2)\to\cdots\to 13\to 12\to 10\to 11=y$ are the xy-path of length $l=d+2,d+4,\cdots,3d-2,\cdots,d^2-2$ in $Q_2(d)$.

Assuming the theorem holds for any k with $2\leqslant k<n$. Let $x=x_1x_2\cdots x_n$ and $y=y_1y_2\cdots y_n$ be any two vertices with distance m in $Q_n(d)$ and let l be

an integer with $m \leqslant l \leqslant d^n-1$ and $l-m$ is an even number. Let $Q_n(d)=0Q_{n-1}(d)\Theta 1Q_{n-1}(d)\Theta\cdots\Theta(d-1)Q_{n-1}(d)$.

Case 1 $m<n$

By the vertex-transitivity of $Q_n(d)$[3], without loss of generality, we can assume $x,y\in V(0Q_{n-1}(d))$. By the induction hypothesis, there is an xy-path of length l in $Q_n(d)$, where $m\leqslant l\leqslant d^{n-1}-1$.

Assuming $d^{n-1}\leqslant l\leqslant 2\times d^{n-1}-1$. Let P_0 be the longest xy-path in $0Q_{n-1}(d)$, the length of P_0 is l_{P_0} and $l_{P_0}-m$ is an even number. We have $l_{P_0}=d^{n-1}-1$ if m is odd and $l_{P_0}=d^{n-1}-2$ if m is even. Let $l_1=l-l_{P_0}-1$. Then l_1 is odd and less than d^{n-1}. Let uv be any edge in P_0, and $u,v\in 0Q_{n-1}(d)$, $u\neq x$, $u\neq y$, $v\neq x$, $v\neq y$. Then $P_0=P_{0_{xu}}+uv+P_{0_{vy}}$. Let u' and v' be neighbors of u and v in $1Q_{n-1}(d)$. By the induction hypothesis, there is a $u'v'$-path P_1 of length l_1 in $1Q_{n-1}(d)$. Then $P_{0_{xu}}+uu'+P_1+v'v+P_{0_{vy}}$ is an xy-path of length l in $0Q_{n-1}(d)\Theta 1Q_{n-1}(d)$, this is also an xy-path of length l in $Q_n(d)$.

Assuming $2\times d^{n-1}\leqslant l\leqslant 3\times d^{n-1}-1$. Let P_{01} be the longest xy-path in $0Q_{n-1}(d)\Theta 1Q_{n-1}(d)$, the length of P_{01} is $l_{P_{01}}$ and $l_{P_{01}}-m$ is an even number. We have $l_{P_{01}}=2\times d^{n-1}-1$ if m is odd and $l_{P_{01}}=2\times d^{n-1}-2$ if m is even. Let $l_2=l-l_{P_{01}}-1$. Then l_2 is odd and less than d^{n-1}. Let u_1v_1 be any edge in P_{01}, and $u_1,v_1\in 1Q_{n-1}(d)$, $u_1\neq u'$, $u_1\neq v'$, $v_1\neq u'$, $v_1\neq v'$. Then $P_{01}=P_{01_{xu_1}}+u_1v_1+P_{01_{v_1y}}$. Let u'_1 and v_1' be neighbors of u_1 and v_1 in $2Q_{n-1}(d)$. By the induction hypothesis, there is an u'_1v_1'-path P_2 of length l_2 in $2Q_{n-1}(d)$. Then $P_{01_{xu_1}}+u_1u_1'+P_2+v_1'v_1+P_{01_{v_1y}}$ is an xy-path of length l in $0Q_{n-1}(d)\Theta 1Q_{n-1}(d)\Theta 2Q_{n-1}(d)$, this is also an xy-path of length l in $Q_n(d)$.

…,…,…

Assuming $(d-1)\times d^{n-1}\leqslant l\leqslant d^n-1$. Let $P_{01\cdots(d-2)}$ be the longest xy-path in $0Q_{n-1}(d)\Theta 1Q_{n-1}(d)\Theta\cdots\Theta(d-2)Q_{n-1}(d)$, the length of $P_{01\cdots(d-2)}$ is $l_{P_{01\cdots(d-2)}}$ and $l_{P_{01\cdots(d-2)}}-m$ is an even number. We have $P_{01\cdots(d-2)}=(d-1)\times d^{n-1}-1$ if m is odd and $P_{01\cdots(d-2)}=(d-1)\times d^{n-1}-2$ if m is even. Let $l_{d-1}=l-l_{P_{01\cdots(d-2)}}-1$. Then l_{d-1} is odd and less than d^{n-1}. Let $u_{d-2}v_{d-2}$ be any edge in $P_{01\cdots(d-2)}$, and $u_{d-2},v_{d-2}\in(d-2)Q_{n-1}(d)$, $u_{d-2}\neq u'_{d-3}$, $u_{d-2}\neq v'_{d-3}$, $v_{d-2}\neq u'_{d-3}$, $v_{d-2}\neq v'_{d-3}$. Then $P_{01\cdots(d-2)}=P_{01\cdots(d-2)_{xu_{d-2}}}+u_{d-2}v_{d-2}+P_{01\cdots(d-2)_{v_{d-2}y}}$. Let u'_{d-2} and v'_{d-2} be neighbors of u_{d-2} and v_{d-2} in $(d-1)Q_{n-1}(d)$. By the induction hypothesis, there is an $u'_{d-2}v'_{d-2}$-path P_{d-1} of length l_{d-1} in $(d-1)Q_{n-1}(d)$. Then $P_{01\cdots(d-2)_{xu_{d-2}}}+u_{d-2}u'_{d-2}+P_{d-1}+v'_{d-2}v_{d-2}+P_{01\cdots(d-2)_{v_{d-2}y}}$ is an xy-path of length l in $Q_n(d)$.

Case 2 $m=n$

By the vertex-transitivity of $Q_n(d)$[3], without loss of generality, we can assume $x\in V(0Q_{n-1}(d))$, $y\in V(1Q_{n-1}(d))$. Let v be a neighbor of y in $1Q_{n-1}(d)$, u be the neighbor of v in $0Q_{n-1}(d)$. Then $D(Q_{n-1}(d);x,u)=n-2$.

If $n\leqslant l\leqslant d^{n-1}+1$. By the induction hypothesis, there is an xu-path P of length $l-2$ in $0Q_{n-1}(d)$, Then $P+uv+vy$ is an xy-path of length l in $Q_n(d)$.

If $d^{n-1}+2\leqslant l\leqslant 2\times d^{n-1}-1$. Let P_0 be the longest xu-path in $0Q_{n-1}(d)$, the length of P_0 is l_{P_0} and $l_{P_0}-m$ is an even number. We have $l_{P_0}=d^{n-1}-1$ if m is odd and $l_{P_0}=d^{n-1}-2$ if m is even. Let $l_1=l-l_{P_0}-1$. Then l_1 is odd and less than d^{n-1}. By the induction hypothesis, there is a vy-path P_1 of length l_1 in $1Q_{n-1}(d)$. Then P_0+uv+P_1 is an xy-path of length l in $0Q_{n-1}(d)\Theta 1Q_{n-1}(d)$, this is also an xy-path of length l in $Q_n(d)$.

If $2\times d^{n-1}\leqslant l\leqslant 3\times d^{n-1}-1$. Let P_{01} be the longest xy-path in $0Q_{n-1}(d)\Theta 1Q_{n-1}(d)$, the length of P_{01} is $l_{P_{01}}$ and $l_{P_{01}}-m$ is an even number. We have $l_{P_{01}}=2\times d^{n-1}-1$ if m is odd and $l_{P_{01}}=2\times d^{n-1}-2$ if m is even. Let $l_2=l-l_{P_{01}}-1$. Then l_2 is odd and less than d^{n-1}. Let u_1v_1 be any edge in P_{01}, and $u_1,v_1\in 1Q_{n-1}(d)$, $u_1\neq v$, $u_1\neq y$, $v_1\neq v$, $v_1\neq y$. Then $P_{01}=P_{01_{xu_1}}+u_1v_1+P_{01_{v_1y}}$. Let u'_1 and v'_1 be neighbors of u_1 and v_1 in $2Q_{n-1}(d)$. By the induction hypothesis, there is an $u'_1v'_1$-path P_2 of length l_2 in $2Q_{n-1}(d)$. Then $P_{01_{xu_1}}+u_1u'_1+P_2+v'_1v_1+$

$P_{01_{v_1 y}}$ is an xy-path of length l in $0Q_{n-1}(d)\Theta 1Q_{n-1}(d)\Theta 2Q_{n-1}(d)$, this is also an xy-path of length l in $Q_n(d)$.

The rest of the proof is similar to Case 1.

By the induction principle, the theorem follows.

Applying Theorem 1, we have

Corollary 1 For any $n\geqslant 2$, every edge of $Q_n(d)$ ($d\geqslant 2$, d is an even number) lies on a cycle of every even length from 4 to d^n.

Applying Theorem 1. If $d=2$, we have

Corollary 2[1,3] Let x and y be any two vertices in Q_n ($n\geqslant 2$) and l be any integer with $D(Q_n;x,y)\leqslant l\leqslant 2^n-1$. If $l-D(Q_n;x,y)$ is an even number, then there is an xy-path of length l in Q_n.

Let F be a set of faulty vertices in $Q_n(d)$.

Lemma 3 For any subset F of $V(Q_2(d))$ ($d\geqslant 4$, d is an even number) with $|F|\leqslant 1$, every edge of $Q_2(d)-F$ lies on a fault-free k-cycle, $k=4,6,\cdots,d^2-2|F|$.

Proof In this article, the operation is modulo d. By corollary 1, we only consider $|F|=1$. Since $Q_2(d)$ is vertex-transitive[3], without loss of generality, we may assume that the faulty vertex is $w=00$. Let $e=(u,v)=(x_1^* x_2^*, x_1^* x_2^{**})$ be a fault-free edge of $Q_2(d)$. We may assume that $x_1^*\neq 0$ ($x_1^*=0$ is similar), $(x_1^* x_2^*, x_1^*(x_2^*+1),\cdots,x_1^*(x_2^*+2i), x_1^* x_2^{**}, x_1^* x_2^*)$ $(i=1,2,\cdots,\frac{d-2}{2}$; If $x_2^*+j=x_2^{**}$, $j=1,2,\cdots,2i$, x_2^*+j is replaced by x_2^*+2i+1 is a $(2i+2)$-cycle and contains the edge e.

$(x_1^* x_2^*, x_1^*(x_2^*+1),\cdots,x_1^*(x_2^*+d-2), x_1^*(x_2^*+d-1),\cdots,(x_1^*+k)(x_2^*+k\times(d-1)), (x_1^*+k)(x_2^*+k\times(d-1)+1),\cdots,(x_1^*+k)(x_2^*+k\times(d-1)+2i),(x_1^*+k)x_2^*, x_1^* x_2^{**}, x_1^* x_2^*)$ $(k=1,2,\cdots,d-2; i=0,1,\cdots,\frac{d-2}{2}$; If $x_1^*+k=0$, x_1^*+k is replaced by x_1^*+k+1. If $x_2^{**}=(x_2^*+k\times(d-1)+j)\bmod d$, $j=1,2,\cdots,2i$, $x_2^*+k\times(d-1)+j$ is replaced by $x_2^*+k\times(d-1)+2i+1)$ is a $(k\times d+2i+2)$-cycle and contains the edge e.

We may assume that $x_2^{**}\neq 0$ ($x_2^{**}=0$ similar), $(x_1^* x_2^*, x_1^*(x_2^*+1),\cdots,x_1^*(x_2^*+d-2), x_1^*(x_2^*+d-1),(x_1^*+1)(x_2^*+d-1),(x_1^*+1)(x_2^*+d),\cdots,(x_1^*+1)(x_2^*+2\times(d-1)),\cdots,(x_1^*+d-2)(x_2^*+(d-2)(d-1)),(x_1^*+d-2)(x_2^*+(d-2)(d-1)+1),\cdots,(x_1^*+d-2)(x_2^*+(d-1)(d-1)),0(x_2^*+(d-1)(d-1)),0(x_2^*+(d-1)(d-1)+1),\cdots,0(x_2^*+(d-1)(d-1)+2i),0x_2^{**}, x_1^* x_2^{**}, x_1^* x_2^*)$, $(i=0,1,\cdots,\frac{d-2}{2}$; If $0=(x_2^*+(d-1)\times(d-1)+j)\bmod d$, $j=1,2,\cdots,2i$, $x_2^*+(d-1)\times(d-1)+j$ is replaced by $x_2^*+(d-1)\times(d-1)+2i+1)$ is a $((d-1)\times d+2i+2)$ and contains the edge e.

Lemma 4 For any subset F of $V(Q_3(d))$ ($d\geqslant 2$, d is an even number) with $|F|\leqslant 1$, every edge of $Q_3(d)-F$ lies on a fault-free k-cycle, $k=4,6,\cdots,d^3-2|F|$.

Proof By Corollary 1, we only consider $|F|=1$. Since $Q_3(d)$ is vertex-transitive[3], without loss of generality, we may assume that the faulty vertex is $w=000$. Let $e=(u,v)$ be a fault-free edge of $Q_3(d)$. By Lemma 2, $Q_3(d)$ can be partitioned into $dQ_2(d)$, denoted by $Q_2^i(d)$, $0\leqslant i\leqslant d-1$; $e\in Q_{n-1}^m(d)$ ($m\in\{1,2,\cdots,d-1\}$). Without loss of generality, we may assume that $Q_3(d)$ is partitioned along dimension j ($j\in\{1,2,3\}$) into $dQ_2(d)$, $e\in Q_2^1(d)$ (If $e\notin Q_2^1(d)$ is similar). By Corollary 1, there is a fault-free even k-cycle in $Q_2^1(d)$ containing the edge e where $4\leqslant k\leqslant d^2$. Thus, the cycle of every even length from 4 to d^2 containing the edge e in $Q_3(d)$ can be found in $Q_2^1(d)$. Let C_1^* be a fault-free even d^2-cycle containing the edge e in $Q_2^1(d)$. Because $d^2\geqslant 4$, therefore, C_1^* has an edge (u_1,v_1), $(u_1,v_1)\neq e$, the cycle C_1^* can be represented as $(u_1,v_1,P_1[v_1,u_1],u_1)$ where e lies on the path $P_1[v_1,u_1]$.

$u_1^{j(2)}\in Q_2^2(d)$, $v_1^{j(2)}\in Q_2^2(d)$, $h(u_1,v_1)=1$, $h(u_1,u_1^{j(2)})=1$, $h(v_1,v_1^{j(2)})=1$, $h(u_1^{j(2)},v_1^{j(2)})=1$. By Corollary 1, there are even cycles with lengths from 4 to d^2 inclusive in $Q_2^2(d)$ that each cycle contains the edge $(u_1^{j(2)},v_1^{j(2)})$. Let $C_{l_2}=(v_1^{j(2)},u_1^{j(2)},$

$P_2[u_1^{j(2)},v_1^{j(2)}],v_1^{j(2)})$ be an even l_2-cycle containing the edge $(u_1^{j(2)},v_1^{j(2)})$ in $Q_2^2(d)$ where $4\leqslant l_2\leqslant d^2$. Merging the two cycles C_1^* and C_{l_2} as well as the two edge $(u_1,u_1^{j(2)})$ and $(v_1,v_1^{j(2)})$, we can construct a fault-free even cycle $C_{12}=(v_1,P_1[v_1,u_1],u_1,u_1^{j(2)},P_2[u_1^{j(2)},v_1^{j(2)}],v_1^{j(2)},v_1)$ which contains e. Obviously, $l(C_{12})=l(P_1[v_1,u_1])+l(P_2[u_1^{j(2)},v_1^{j(2)}])+2$ where $l(P_1[v_1,u_1])=d^2-1$ and $l(P_2[u_1^{j(2)},v_1^{j(2)}])=1,3,\cdots,d^2-1$. Therefore, C_{12} is an even cycle of length from d^2+2 to $2d^2$ and contains the edge e.

Let $C_{12\cdots i}^*(i=2,3,\cdots,d-2)$ be a fault-free even $i\times d^2$-cycle containing the edge e. $C_{12\cdots i}^*$ has an edge $(u_i,v_i),(u_i,v_i)\notin\{e,(u_1,v_1),\cdots,(u_{i-1},v_{i-1})\}$, the cycle $C_{12\cdots i}^*$ can be represented as $(u_i,v_i,P_{12\cdots i}[v_i,u_i],u_i)$ where e lies on the path $P_{12\cdots i}[v_i,u_i]$. $u_i^{j(i+1)}\in Q_2^{i+1}(d),v_i^{j(i+1)}\in Q_2^{i+1}(d),h(u_i,v_i)=1$, $h(u_i,u_i^{j(i+1)})=1$, $h(v_i,v_i^{j(i+1)})=1$, $h(u_i^{j(i+1)},v_i^{j(i+1)})=1$. By Corollary 1, there are even cycles with lengths from 4 to d^2 inclusive in $Q_2^{i+1}(d)$ that each cycle contains the edge $(u_i^{j(i+1)},v_i^{j(i+1)})$. Let $C_{l_{i+1}}=(v_i^{j(i+1)},u_i^{j(i+1)},P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}],v_i^{j(i+1)})$ be an even l_{i+1}-cycle containing the edge $(u_i^{j(i+1)},v_i^{j(i+1)})$ in $Q_2^{i+1}(d)$ where $4\leqslant l_{i+1}\leqslant d^2$. Merging the two cycles $C_{12\cdots i}^*$ and $C_{l_{i+1}}$ as well as the two edge $(u_i,u_i^{J(i+1)})$ and $(v_i,v_i^{J(i+1)})$, we can construct a fault-free even cycle $C_{12\cdots(i+1)}=(v_i,P_{12\cdots i}[v_i,u_i],u_i,u_i^{j(i+1)},P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}],v_i^{j(i+1)},v_i)$ which contains e. Obviously, $l(C_{12\cdots(i+1)})=l(P_{12\cdots i}[v_i,u_i])+l(P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}])+2$ where $l(P_{12\cdots i}[v_i,u_i])=i\times d^2-1$ and $l(P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}])=1,3,\cdots,d^2-1$. Therefore, $C_{12\cdots(i+1)}$ is an even cycle of length from $i\times d^2+2$ to $(i+1)\times d^2$ and contains the edge e.

Let $C_{12\cdots(d-1)}^*$ be a fault-free even $(d-1)\times d^2$-cycle containing the edge e. $C_{12\cdots(d-1)}^*$ has an edge $(u_{d-1},v_{d-1}),(u_{d-1},v_{d-1})\notin\{e,(u_1,v_1),\cdots,(u_{d-2},v_{d-2})\}$, the cycle $C_{12\cdots(d-1)}^*$ can be represented as $(u_{d-1},v_{d-1},P_{12\cdots(d-1)}[v_{d-1},u_{d-1}],u_{d-1})$ where e lies on the path $P_{12\cdots(d-1)}[v_{d-1},u_{d-1}]$. $u_{d-1}^{j(0)}\in Q_2^0(d)-F,v_{d-1}^{j(0)}\in Q_2^0(d)-F,h(u_{d-1},v_{d-1})=1$, $h(u_{d-1},u_{d-1}^{j(0)})=1$, $h(v_{d-1},v_{d-1}^{j(0)})=1$, $h(u_{d-1}^{j(0)},v_{d-1}^{j(0)})=1$. By Lemma 3, there are even cycles with lengths from 4 to $d^2-2|F|$ inclusive in $Q_2^0(d)-F$ that each cycle contains the edge $(u_{d-1}^{j(0)},v_{d-1}^{j(0)})$. Let $C_{l_0}=(v_{d-1}^{j(0)},u_{d-1}^{j(0)},P_0[u_{d-1}^{j(0)},v_{d-1}^{j(0)}],v_{d-1}^{j(0)})$ be an even l_0-cycle containing the edge $(u_{d-1}^{j(0)},v_{d-1}^{j(0)})$ in $Q_2^0(d)-F$ where $4\leqslant l_0\leqslant d^2-2|F|$. Merging the two cycles $C_{12\cdots(d-1)}^*$ and C_{l_0} as well as the two edge $(u_{d-1},u_{d-1}^{J(0)})$ and $(v_{d-1},v_{d-1}^{J(0)})$, we can construct a fault-free even cycle $C_{12\cdots(d-1)0}=(v_{d-1},P_{12\cdots(d-1)}[v_{d-1},u_{d-1}],u_{d-1},u_{d-1}^{j(0)},P_0[u_{d-1}^{j(0)},v_{d-1}^{j(0)}],v_{d-1}^{j(d-1)},v_{d-1})$ which contains e. Obviously, $l(C_{12\cdots(d-1)0})=l(P_{12\cdots(d-1)}[v_{d-1},u_{d-1}])+l(P_0[u_{d-1}^{j(0)},v_{d-1}^{j(0)}])+2$ where $l(P_{12\cdots(d-1)}[v_{d-1},u_{d-1}])=(d-1)\times d^2-1$ and $l(P_0[u_{d-1}^{j(0)},v_{d-1}^{j(0)}])=1,3,\cdots,d^2-1-2|F|$. Therefore, $C_{12\cdots(d-1)0}$ is an even cycle of length from $(d-1)\times d^2+2$ to $d^3-2|F|$ and contains the edge e.

Similar to Lemma 4, we have

Theorem 2 Let $n\geqslant3$ be an integer and $Q_n(d)$ ($d\geqslant2$, d is an even number) has exactly one faulty vertex. Then, every fault-free edge of $Q_n(d)$ lies on a fault-free cycle of every even length from 4 to d^n-2.

Theorem 3 Let $n\geqslant3$ be an integer. For any subset F of $V(Q_n(d))$ ($d\geqslant2$, d is an even number) with $|F|=f_v\leqslant n-2$, every edge of $Q_n(d)-F$ lies on a cycle of every even length from 4 to d^n-2f_v.

Proof We prove this theorem by induction on n. By Lemma 4, Theorem 3 holds for $n=3$. Assuming that the theorem is true for every integer k ($3\leqslant k\leqslant n$). Let F be a subset of $V(Q_{k+1}(d))$ and $|F|=f_v$. By Corollary 1 and Theorem 2, Theorem 3 holds for $f_v\leqslant1$. Thus, we only consider the case of $2\leqslant f_v\leqslant n-2$.

Let w and z be two distinct faulty vertices. By Lemma 1, $Q_{k+1}(d)$ can be partitioned along dimension j ($j\in\{1,2,\cdots,k+1\}$ into d copies $Q_k(d)$, denoted by $Q_k^i(d)(i=0,1,2,\cdots,d-1)$, $w\in Q_k^l(d)$, $z\in Q_k^m(d)$ ($l,m\in\{0,1,2,\cdots,d-1\},l\neq m$). Let $f_i=|F\cap V(Q_k^i(d))|$. $i=0,1,2,\cdots,d-1$, i.e., $f_v=\sum_{i=0}^{d-1}f_i$. Therefore, $f_i\leqslant k-2,i=0,1,2,\cdots,d-$

1. Let $e=(u,v)$ be a fault-free edge of $Q_{k+1}(d)-F$. In order to prove this theorem, we establish every even l-cycle containing e where $4\leqslant l\leqslant d^{k+1}-2f_v$.

Case 1: $e\in E(Q_k^0(d))\cup E(Q_k^1(d))\cup\cdots\cup E(Q_k^{d-1}(d))$, i.e., e lies on $Q_k^i(d)(i=0,1,2,\cdots,d-1)$. We only consider that $e\in E(Q_k^0(d))(e\notin E(Q_k^0(d))$ is similar).

Since $f_0\leqslant k-2$, by induction hypothesis, there is a fault-free even l_0-cycle in $Q_k^0(d)$ containing the edge e where $4\leqslant l_0\leqslant d^k-2f_0$. Thus, the cycle of every even length from 4 to d^k-2f_0 containing the edge e in $Q_{k+1}(d)$ can be found in $Q_k^0(d)$. Let $C_{l_0^*}$ be a fault-free even l_0^*-cycle containing the edge e in $Q_k^0(d)$ where $l_0^*=d^k-2f_0$. One can observe that there are at least $\frac{1}{2}\times d^k-f_0-1$ disjoint edges such that each of them differs with e in the cycle $C_{l_0^*}$. Since $k\geqslant 3$ and $\sum_{i=0}^{k+1}f_i\leqslant k-1$, $\frac{1}{2}\times d^k-f_0-1>\sum_{i=1}^{k+1}f_i$. Therefore, $C_{l_0^*}$ has an edge (u_0,v_0), $(u_0,v_0)\neq e$, $u_0^{j(m)}$ is a fault-free vertex in $Q_k^m(d)$, $v_0^{j(m)}$ is a fault-free vertex in $Q_k^m(d)(m\in\{1,2,\cdots,d-1\}$, $h(u_0,u_0^{j(m)})=1$, $h(v_0,v_0^{j(m)})=1$. We may assume that $m=1$ ($m\neq 1$ is similar), i.e., $u_0^{j(1)}$ is a fault-free vertex in $Q_k^1(d)$, $v_0^{j(1)}$ is a fault-free vertex in $Q_k^1(d)$. The cycle $C_{l_0^*}$ can be represented as $(u_0,v_0,P_0[v_0,u_0],u_0)$ where e lies on the path $P_0[v_0,u_0]$.

Since $f_1\leqslant k-2$, by induction hypothesis, there are even cycles with lengths from 4 to d^k-2f_1 in $Q_k^1(d)$ that each cycle contains the edge $(u_0^{j(1)},v_0^{j(1)})$, Let $C_{l_1}=(v_0^{j(1)},u_0^{j(1)},P_1[u_0^{j(1)},v_0^{j(1)}],v_0^{j(1)})$ be an even l_1-cycle containing the edge $(u_0^{j(1)},v_0^{j(1)})$ in $Q_k^1(d)$ where $4\leqslant l_1\leqslant d^k-2f_1$. Merging the two cycles $C_{l_0^*}$ and C_{l_1} as well as the two edges $(u_0,u_0^{j(1)})$ and $(v_0,v_0^{j(1)})$, we can construct a fault-free even cycle $C_{01}=(v_0,P_0[v_0,u_0],u_0,u_0^{j(1)},P_1[u_0^{j(1)},v_0^{j(1)}],v_0^{j(1)},v_0)$ which contains e. Obviously, $l(C_{01})=l(P_0[v_0,u_0])+l(P_1[u_0^{j(1)},v_0^{j(1)}])+2$ where $l(C_{01})=d^k-2f_0-1$, and $l(P_1[u_0^{j(1)},v_0^{j(1)}])=1,3,\cdots,d^k-2f_1-1$. Therefore, the cycle C_{01} is of length from d^k-2f_0+2 to $2\times d^k-2(f_0+f_1)$ and contains the edge e.

Let $C_{012\cdots i}^*(i=1,2,\cdots,d-2)$ be a fault-free even $((i+1)\times d^k-2\sum_{a=0}^{i}f_a)$-cycle containing the edge e. One can observe that there are at least $\frac{1}{2}\times(i+1)d^k-\sum_{a=0}^{i}f_a-1$ disjoint edges such that each of them differs with e in the cycle $C_{012\cdots i}^*$. Since $k\geqslant 3$ and $\sum_{a=0}^{k+1}f_a\leqslant k-1$, $\frac{1}{2}\times(i+1)d^k-\sum_{a=0}^{i}f_a-i>\sum_{a=i+1}^{k+1}f_a$. Therefore, $C_{012\cdots i}^*$ has an edge (u_i,v_i), $(u_i,v_i)\in\{e,(u_1,v_1),\cdots,(u_{i-1},v_{i-1})\}$, $u_i^{j(m)}$ is a fault-free vertex in $Q_k^m(d)$, $v_i^{j(m)}$ is a fault-free vertex in $Q_k^m(d)(m\in\{i+1,i+2,\cdots,d-1\})$, $h(u_i,u_i^{j(m)})=1$, $h(v_i,v_i^{j(m)})=1$. We may assume that $m=i+1$ ($m\neq i+1$ is similar), i.e., $u_i^{j(i+1)}$ is a fault-free vertex in $Q_k^{k+1}(d)$, $v_i^{j(i+1)}$ is a fault-free vertex in $Q_k^{k+1}(d)$. The cycle $C_{012\cdots i}^*$ can be represented as $(u_i,v_i,P_{012\cdots i}[v_i,u_i],u_i)$ where e lies on the path $P_{012\cdots i}[v_i,u_i]$.

Since $f_{i+1}\leqslant k-2$, by induction hypothesis, there are even cycles with lengths from 4 to d^k-2f_{i+1} in $Q_k^{i+1}(d)$ that each cycle contains the edge $(u_i^{j(i+1)},v_i^{j(i+1)})$. Let $C_{l_{i+1}}=(v_i^{j(i+1)},u_i^{j(i+1)},P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}],v_i^{j(i+1)})$ be an even l_{i+1}-cycle containing the edge $(u_i^{j(i+1)},v_i^{j(i+1)})$ in $Q_k^{i+1}(d)$ where $4\leqslant l_{i+1}\leqslant d^k-2f_{i+1}$. Merging the two cycles $C_{012\cdots i}^*$ and $C_{l_{i+1}}$ as well as the two edges $(u_i,u_i^{j(i+1)})$ and $(v_i,v_i^{j(i+1)})$, we can construct a fault-free even cycle $C_{01\cdots i(i+1)}=(v_i,P_{01\cdots i}[v_i,u_i],u_i,u_i^{j(i+1)},P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}],v_i^{j(i+1)},v_i)$ which contains e. Obviously, $l(C_{01\cdots i(i+1)})=l(P_{01\cdots i}[v_i,u_i])+l(P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}])+2$ where $l(P_{01\cdots i}[v_i,u_i])=(i+1)\times d^k-2\sum_{a=0}^{i}f_a-1$, and $l(P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}])=1,3,\cdots,d^k-2f_{i+1}-1$. Therefore, the cycle $C_{01\cdots i(i+1)}$ is of length from $(i+1)\times d^k-2\sum_{a=0}^{i}f_a+2$ to $(i+2)\times d^k-2\sum_{a=0}^{i+1}f_a$ and contains the edge e.

Case 2: $e\notin E(Q_k^0(d))\cup E(Q_k^1(d))\cup\cdots\cup E(Q_k^{d-1}(d))$, i.e., $u\in Q_k^l(d)(l\in\{0,1,\cdots,d-1\})$,

$v \in Q_k^m(d)$ $(m \in \{0,1,\cdots,d-1\})$, $l \neq m$, e is an edge of dimension j and $v = u^{j(a)}$ $(j \in \{1,2,\cdots,k+1\}$, $a \in \{0,1,\cdots,d-1\})$.

We assume that $u \in Q_k^0(d)$ and $v \in Q_k^1(d)$ (If $u \notin Q_k^0(d)$ or $v \notin Q_k^1(d)$ is similar). Since $f_v \leqslant (k+1)-2 = k-1$, there is an integer i $(i \in \{1,2,\cdots,k+1\})$, $i \neq j$, such that $u^{i(a)}$ and $v^{i(a)}$ $(a \in \{0,1,\cdots,d-1\})$ are fault-free. Thus, $(u, u^{i(a)}, v^{i(a)}, v, u)$ is a fault-free 4-cycle containing the edge e. Noting that u and $u^{i(a)}$ (respectively, v and $v^{i(a)}$) are adjacent in $Q_k^0(d)$ (respectively, $Q_k^1(d)$). Since $f_0 \leqslant k-2$ and $f_1 \leqslant k-2$, by induction hypothesis, there is an even l_0-cycle in $Q_k^0(d)$ containing the edge $(u, u^{i(a)})$ such as $C_{l_0} = (u, u^{i(a)}, P_0[u^{i(a)}, u], u)$ and there is an even l_1-cycle in $Q_k^1(d)$ containing the edge $(v, v^{i(a)})$ such as $C_{l_1} = (v^{i(a)}, v, P_1[v, v^{i(a)}], v^{i(a)})$ where $4 \leqslant l_0 \leqslant d^k - 2f_0$ and $4 \leqslant l_1 \leqslant d^k - 2f_1$. Combining the 4-cycle $(u, u^{i(a)}, v^{i(a)}, v, u)$ and a 4-cycle containing $(u, u^{i(a)})$ in $Q_k^0(d)$, the desired 6-cycle can be obtained. Merging the two cycles C_{l_0} and C_{l_1} as well as the two edges (u, v) and $(u^{i(a)}, v^{i(a)})$, we can construct a fault-free even cycle $C_{01} = (u, v, P_1[v, v^{i(a)}], v^{i(a)}, u^{i(a)}, P_0[u^{i(a)}, u], u)$ which contains e. Obviously, $l(C_{01}) = l(P_1[v, v^{i(a)}]) + l(P_0[u^{i(a)}, u]) + 2$ where $l(P_0[u^{i(a)}, u]) = 3,5,\cdots,d^k - 2f_0 - 1$ and $l(P_1[v, v^{i(a)}]) = 3,5,\cdots,d^k - 2f_1 - 1$. This implies that $8 \leqslant l(C_{01}) \leqslant 2 \times d^k - 2(f_0 + f_1)$, $l(C_{01})$ is even and C_{01} contains the edge e.

Let $C_{012\cdots i}^*$ $(i = 1,2,\cdots,d-2)$ be a fault-free even $((i+1) \times d^k - 2\sum_{a=0}^{i} f_a)$-cycle containing the edge e. Similar to Case 1, we can construct a fault-free even cycle $C_{01\cdots i(i+1)} = (v_i, P_{01\cdots i}[v_i, u_i], u_i, u_i^{j(i+1)}, P_{i+1}[u_i^{j(i+1)}, v_i^{j(i+1)}], v_i^{j(i+1)}, v_i)$ which contains e. The cycle $C_{01\cdots i(i+1)}$ is of length from $(i+1) \times d^k - 2\sum_{a=0}^{i} f_a + 2$ to $(i+2) \times d^k - 2\sum_{a=0}^{i} f_a$ and contains the edge e.

Since $|F| \leqslant n-2$ and the degree of any vertex of $Q_n(d)$ is $n(d-1)$, any fault-free vertex of $Q_n(d)$ has at least $n(d-2)+2$ fault-free neighbors. Thus, every fault-free vertex can be incident by a fault-free edge. Therefore, we have

Corollary 3 Let $n \geqslant 3$ be an integer. For any subset F of $V(Q_n(d))$ $(d \geqslant 2, d$ is an even number) with $|F| \leqslant n-2$, every vertex of $Q_n(d) - F$ lies on a fault-free cycle of every even length from 4 to $d^n - 2|F|$.

Applying Theorem 3. If $d = 2$, we have

Corollary 4[2] Assuming that $n \geqslant 3$. For any subset F of $V(Q_n(d))$ with $|F| = f_v \leqslant n-2$, every edge of $Q_n(d) - F$ lies on a cycle of every even length from 4 to $2^n - 2f_v$.

Applying Corollary 4. We have

Corollary 5[2] Let $n \geqslant 3$ be an integer. For any subset F of $V(Q_n(d))$ with $|F| \leqslant n-2$, every vertex of $Q_n - F$ lies on a fault-free cycle of every even length from 4 to $2^n - 2|F|$.

3 *d* is an odd number

Theorem 4 Let x and y be any two vertices in $Q_n(d)$ $(n \geqslant 2)$ and l be any integer with $D(Q_n(d); x, y) \leqslant l \leqslant d^n - 1$. If d is an odd number, $l - D(Q_n(d); x, y)$ is an even number, then there is an xy-path of length l in $Q_n(d)$. Moreover, if $D(Q_n(d); x, y) = 1$, there is an xy-path of length $l = d^n - 1$ in $Q_n(d)$.

Proof Let $D(Q_n(d); x, y) = m$. The proof is based on the recursive structure of $Q_n(d)$ by induction on $n \geqslant 2$.

When $n = 2$, if $D(Q_n(d); x, y) = 1$. By the vertex-transitivity of $Q_2(d)$[3], without loss of generality, we can assume $x = 00$, $y = 01$.

$x = 00 \to 01 = y$, $x = 00 \to 02 \to 03 \to 01 = y$, $x = 00 \to 02 \to 03 \to 04 \to 05 \to 01 = y$, $\cdots$, $x = 00 \to 02 \to 03 \to 04 \to 05 \to \cdots \to 0(d-3) \to 0(d-2) \to 01 = y$ are the xy-path of length $l = 1,3,5,\cdots,d-2$ in $Q_2(d)$.

$x = 00 \to 10 \to 12 \to 02 \to 03 \to 04 \to 05 \to \cdots \to 0(d-3) \to 0(d-2) \to 01 = y$, $x = 00 \to 10 \to 20 \to 22 \to 12 \to 02 \to 03 \to 04 \to 05 \to \cdots \to 0(d-3) \to 0(d-2) \to 01 = y$, $\cdots$.

$x = 00 \to 10 \to 20 \to 30 \to 40 \to \cdots \to (d-2)0 \to (d-1)0 \to (d-1)2 \to (d-2)2 \to \cdots \to 22 \to 12 \to 02 \to 03 \to 13 \to 23 \to \cdots \to (d-2)3 \to (d-1)3 \to (d-1)4 \to (d-2)4 \to \cdots \to 24 \to 14 \to 04 \to \cdots \to 0(d-4) \to 1(d-4) \to 2(d-4) \to \cdots \to (d-2)(d-4) \to (d-1)(d-4) \to$

$(d-1)(d-3)\to(d-2)(d-3)\to\cdots\to2(d-3)\to1(d-3)\to0(d-3)\to0(d-2)\to1(d-2)\to2(d-2)\to\cdots\to(d-2)(d-2)\to(d-1)(d-2)\to(d-1)1\to(d-2)1\to\cdots\to21\to11\to01=y,\cdots$.

$x=00\to10\to20\to30\to40\to\cdots\to(d-2)0\to(d-1)0\to(d-1)2\to(d-2)2\to\cdots\to22\to12\to02\to03\to13\to23\to\cdots\to(d-2)3\to(d-1)3\to(d-1)4\to(d-2)4\to\cdots\to24\to14\to04\to\cdots\to0(d-4)\to1(d-4)\to2(d-4)\to\cdots\to(d-2)(d-4)\to(d-1)(d-4)\to(d-1)(d-3)\to(d-2)(d-3)\to\cdots\to2(d-3)\to1(d-3)\to0(d-3)\to0(d-2)\to1(d-2)\to2(d-2)\to\cdots\to(d-2)(d-2)\to(d-1)(d-2)\to(d-1)1\to(d-1)(d-1)\to(d-2)(d-1)\to(d-2)1\to(d-3)1\to(d-3)(d-1)\to(d-4)(d-1)\to(d-4)1\to\cdots\to21\to2(d-1)\to1(d-1)\to11\to01=y$ are the xy-path of length $l=d,d+2,\cdots,d^2-d-1,\cdots,d^2-2$ in $Q_2(d)$.

$x=00\to10\to20\to30\to40\to\cdots\to(d-2)0\to(d-1)0\to(d-1)2\to(d-2)2\to\cdots\to22\to12\to02\to03\to13\to23\to\cdots\to(d-2)3\to(d-1)3\to(d-1)4\to(d-2)4\to\cdots\to24\to14\to04\to\cdots\to0(d-4)\to1(d-4)\to2(d-4)\to\cdots\to(d-2)(d-4)\to(d-1)(d-4)\to(d-1)(d-3)\to(d-2)(d-3)\to\cdots\to2(d-3)\to1(d-3)\to0(d-3)\to0(d-2)\to1(d-2)\to2(d-2)\to\cdots\to(d-2)(d-2)\to(d-1)(d-2)\to(d-1)1\to(d-1)(d-1)\to(d-2)(d-1)\to(d-2)1\to(d-3)1\to(d-3)(d-1)\to(d-4)(d-1)\to(d-4)1\to\cdots\to21\to2(d-1)\to0(d-1)\to1(d-1)\to11\to01=y$ is the xy-path of length $l=d^2-1$ in $Q_2(d)$.

The rest of the inductive proof are similar to Theorem 1.

Applying Theorem 4, we have

Corollary 6 For any $n\geqslant2$, every edge of $Q_n(d)$ ($d\geqslant3$, d is an odd number) lies on a cycle of every even length from 4 to d^n-1. Moreover, every edge of $Q_n(d)$ lies on a cycle of length d^n.

Similar to Lemma 3. We have

Lemma 5 For any subset F of $V(Q_2(d))$ ($d\geqslant3$, d is an odd number) with $|F|\leqslant1$, every edge of $Q_2(d)-F$ lies on a fault-free k-cycle, $k=4,6,\cdots,d^2-2|F|-1$. Moreover, every edge of $Q_2(d)-F$ lies on a fault-free $(d^2-2|F|)$-cycle.

Similar to Lemma 4, applying Theorem 4 and Lemma 5. We have

Lemma 6 For any subset F of $V(Q_3(d))$ ($d\geqslant3$, d is an odd number) with $|F|\leqslant1$, every edge of $Q_3(d)-F$ lies on a fault-free k-cycle, $k=4,6,\cdots,d^3-2|F|-1$. Moreover, every edge of $Q_3(d)-F$ lies on a fault-free $(d^3-2|F|)$-cycle.

Similar to Lemma 6, we have

Theorem 5 Let $n\geqslant3$ be an integer and $Q_n(d)$ ($d\geqslant3$, d is an odd number) has exactly one faulty vertex. Then, every fault-free edge of $Q_n(d)$ lies on a fault-free cycle of every even length from 4 to d^n-3. Moreover, every fault-free edge of $Q_n(d)$ lies on a fault-free cycle of length d^n-2.

Theorem 6 Let $n\geqslant3$ be an integer. For any subset F of $V(Q_n(d))$ ($d\geqslant3$, d is an odd number) with $|F|=f_v\leqslant n-2$, every edge of $Q_n(d)-F$ lies on a cycle of every even length from 4 to d^n-2f_v-1. Moreover, every edge of $Q_n(d)-F$ lies on a cycle of length d^n-2f_v.

Proof We prove this theorem by induction on n. By Lemma 6, Theorem 6 holds for $n=3$. Assuming that the theorem is true for every integer k ($3\leqslant k\leqslant n$). Let F be a subset of $V(Q_{k+1}(d))$ and $|F|=f_v$. By Corollary 6 and Theorem 5, Theorem 6 holds for $f_v\leqslant1$. Thus, we only consider the case of $2\leqslant f_v\leqslant n-2$.

Let w and z be two distinct faulty vertices. By Lemma 1, $Q_{k+1}(d)$ can be partitioned along dimension j ($j\in\{1,2,\cdots,k+1\}$) into d copies $Q_k(d)$, denoted by $Q_k^i(d)$ ($i=0,1,\cdots,d-1$), $w\in Q_k^l(d)$, $z\in Q_k^m(d)$ ($l,m\in\{0,1,2,\cdots,d-1\}$, $l\neq m$). Let $f_i=|F\cap V(Q_k^i(d))|$, $i=0,1,2,\cdots d-1$, i.e., $f_v=\sum_{i=0}^{d-1}f_i$. Therefore, $f_i\leqslant k-2$, $i=0,1,2,\cdots,d-1$. Let $e=(u,v)$ be a fault-free edge of $Q_{k+1}(d)-F$. In order to prove this theorem, we establish every even l-cycle containing e where $4\leqslant l\leqslant d^{k+1}-2f_v-1$, and a $(d^{k+1}-2f_v)$-cycle containing e.

Case 1: $e\in E(Q_k^0(d))\cup E(Q_k^1(d))\cup\cdots\cup E(Q_k^{d-1}(d))$, i.e., e lies on $Q_k^i(d)$ ($i\in\{0,1,2,\cdots,d-1\}$). We only consider that $e\in E(Q_k^0(d))$ ($e\notin E(Q_k^0(d))$ is similar).

Since $f_0\leqslant k-2$, by induction hypothesis, there is a fault-free even l_0-cycle in $Q_k^0(d)$ containing the edge e where $4\leqslant l_0\leqslant d^k-2f_0-1$, and there exists a fault-free (d^k-2f_0)-cycle in $Q_k^0(d)$ containing the edge e. Thus, the cycle of every even length from 4 to d^k-2f_0-1 containing the edge e in $Q_{k+1}(d)$ can be found in $Q_k^0(d)$. Let $C_{l_0^*}$ ($C_{l_0^{*'}}$) be a fault-free even l_0^*-cycle ($l_0^{*'}$-cycle) containing the edge e in $Q_k^0(d)$ where $l_0^*=d^k-2f_0-1$ ($l_0^{*'}=d^k-2f_0$). One can observe that there are at least $\frac{1}{2}\times(d^k-1)-f_0-1$ disjoint edges such that each of them differs with e in the cycle $C_{l_0^*}$ ($C_{l_0^{*'}}$). Since $k\geqslant 3$ and $\sum_{i=0}^{k+1}f_i\leqslant k-1$, $\frac{1}{2}\times(d^k-1)-f_0-1>\sum_{i=1}^{k+1}f_i$. Therefore, $C_{l_0^*}$ ($C_{l_0^{*'}}$) has an edge (u_0,v_0), $(u_0,v_0)\neq e$, $u_0^{j(m)}$ is a fault-free vertex in $Q_k^m(d)$, $v_0^{j(m)}$ is a fault-free vertex in $Q_k^m(d)$ ($m\in\{1,2,\cdots,d-1\}$), $h(u_0,u_0^{j(m)})=1$, $h(v_0,v_0^{j(m)})=1$. We may assume that $m=1$ ($m\neq 1$ is similar), i.e., $u_0^{j(1)}$ is a fault-free vertex in $Q_k^1(d)$, $v_0^{j(1)}$ is a fault-free vertex in $Q_k^1(d)$. The cycle $C_{l_0^*}$ ($C_{l_0^{*'}}$) can be represented as $(u_0,v_0,P_0[v_0,u_0],u_0)$ where e lies on the path $P_0[v_0,u_0]$.

Since $f_1\leqslant k-2$, by induction hypothesis, there are even cycles with lengths from 4 to d^k-2f_1-1 in $Q_k^1(d)$ that each cycle contains the edge $(u_0^{j(1)},v_0^{j(1)})$, and there is a cycle of length d^k-2f_1 in $Q_k^1(d)$ that the cycle contains the edge $(u_0^{j(1)},v_0^{j(1)})$. Let $C_{l_1}=(v_0^{j(1)},u_0^{j(1)},P_1[u_0^{j(1)},v_0^{j(1)}],v_0^{j(1)})$ be an even l_1-cycle containing the edge $(u_0^{j(1)},v_0^{j(1)})$ in $Q_k^1(d)$ where $4\leqslant l_1\leqslant d^k-2f_1-1$, $C_{l_1'}=(v_0^{j(1)},u_0^{j(1)},P_1[u_0^{j(1)},v_0^{j(1)}],v_0^{j(1)})$ be a (d^k-2f_1)-cycle containing the edge $(u_0^{j(1)},v_0^{j(1)})$ in $Q_k^1(d)$. Merging the two cycles $C_{l_0^*}$ and C_{l_1} as well as the two edges $(u_0,u_0^{j(1)})$ and $(v_0,v_0^{j(1)})$, we can construct a fault-free even cycle $C_{01}=(v_0,P_0[v_0,u_0],u_0,u_0^{j(1)},P_1[u_0^{j(1)},v_0^{j(1)}],v_0^{j(1)},v_0)$ which contains e. Obviously, $l(C_{01})=l(P_0[v_0,u_0])+l(P_1[u_0^{j(1)},v_0^{j(1)}])+2$ where $l(P_0[v_0,u_0])=d^k-2f_0-2$, and $l(P_1[u_0^{j(1)},v_0^{j(1)}])=1,3,\cdots,d^k-2f_1-1$. Therefore, the cycle C_{01} is of length from d^k-2f_0+1 to $2\times d^k-2(f_0+f_1)-2$ and contains the edge e. Merging the two cycles $C_{l_0^{*'}}$ and $C_{l_1'}$ as well as the two edges $(u_0,u_0^{j(1)})$ and $(v_0,v_0^{j(1)})$, we can construct a fault-free even cycle $C_{01}'=(v_0,P_0[v_0,u_0],u_0,u_0^{j(1)},P_1[u_0^{j(1)},v_0^{j(1)}],v_0^{j(1)},v_0)$ which contains e. Obviously, $l(C_{01}')=l(P_0[v_0,u_0])+l(P_1[u_0^{j(1)},v_0^{j(1)}])+2$ where $l(P_0[v_0,u_0])=d^k-2f_0-1$ and $l(P_1[u_0^{j(1)},v_0^{j(1)}])=d^k-2f_1-1$. Therefore, the cycle C_{01}' is $(2\times d^k-2(f_0+f_1))$-cycle and contains the edge e.

Let $C_{012,\cdots,i}^*$ ($i=1,3,\cdots,d-4,d-2$) be a fault-free even $((i+1)\times d^k-2\sum_{a=0}^{i}f_a)$-cycle containing the edge e. One can observe that there are at least $\frac{1}{2}\times(i+1)d^k-\sum_{a=0}^{i}f_a-1$ disjoint edges such that each of them differs with e in the cycle $C_{012,\cdots,i}^*$. Since $k\geqslant 3$ and $\sum_{a=0}^{k+1}f_a\leqslant k-1$, $\frac{1}{2}\times(i+1)d^k-\sum_{a=0}^{i}f_a-1>\sum_{a=i+1}^{k+1}f_a$. Therefore, $C_{012,\cdots,i}^*$ has an edge (u_i,v_i), $(u_i,v_i)\notin\{e,(u_1,v_1),\cdots,(u_{i-1},v_{i-1})\}$, $u_i^{j(m)}$ is a fault-free vertex in $Q_k^m(d)$, $v_i^{j(m)}$ is a fault-free vertex in $Q_k^m(d)$ ($m\in\{i+1,i+2,\cdots,d-1\}$), $h(u_i,u_i^{j(m)})=1$, $h(v_i,v_i^{j(m)})=1$. We may assume that $m=i+1$ ($m\neq i+1$ is similar), i.e., $u_i^{j(i+1)}$ is a fault-free vertex in $Q_k^{i+1}(d)$, $v_i^{j(i+1)}$ is a fault-free vertex in $Q_k^{i+1}(d)$. The cycle $C_{012,\cdots,i}^*$ can be represented as $(u_i,v_i,P_{012\cdots i}[v_i,u_i],u_i)$ where e lies on the $P_{012\cdots i}[v_i,u_i]$.

Since $f_{i+1}\leqslant k-2$, by induction hypothesis, there are even cycles with lengths from 4 to $d^k-2f_{i+1}-1$ in $Q_k^{i+1}(d)$ that each cycle contains the edge $(u_i^{j(i+1)},v_i^{j(i+1)})$, and there is a (d^k-2f_{i+11})-cycle in $Q_k^{i+1}(d)$ that the cycle contains the edge $(u_i^{j(i+1)},v_i^{j(i+1)})$. Let $C_{l_{i+1}}=(v_i^{j(i+1)},u_i^{j(i+1)},P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}],v_i^{j(i+1)})$ be an even l_{i+1}-cycle containing the edge $(u_i^{j(i+1)},v_i^{j(i+1)})$ in $Q_k^{i+1}(d)$ where $4\leqslant l_{i+1}\leqslant d^k-2f_{i+1}-1$, $C_{l_{i+1}'}=(v_i^{j(i+1)},u_i^{j(i+1)},P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}],v_i^{j(i+1)})$ be a (d^k-2f_{i+11})-cycle containing the edge $(u_i^{j(i+1)},v_i^{j(i+1)})$ in $Q_k^{i+1}(d)$. Merging the two cycles $C_{012,\cdots,i}^*$ and $C_{l_{i+1}}$ as well as the two edges $(u_i,u_i^{j(i+1)})$ and $(v_i,v_i^{j(i+1)})$, we can construct a fault-free even cycle $C_{01\cdots i(i+1)}=(v_i,P_{01\cdots i}[v_i,u_i],u_i,u_i^{j(i+1)},P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}],$

$v_i^{j(i+1)}$, v_i) which contains e. Obviously, $l(C_{01\cdots i(i+1)})=l(P_{01\cdots i}[v_i,u_i])+l(P_{i+1}[u_i^{j(i+1)}, v_i^{j(i+1)}]+2$ where $l(P_{01\cdots i}[v_i,u_i])=(i+1)\times d^k-2\sum_{a=0}^{i}f_a-1$, and $l(P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}])=1,3,\cdots,d^k-2f_{i+1}-2$. Therefore, the cycle $C_{01\cdots i(i+1)}$ is of length from $(i+1)\times d^k-2\sum_{a=0}^{i}f_a+2$ to $(i+2)\times d^k-2\sum_{a=0}^{i}f_a-1$ and contains the edge e. Merging the two cycles $C^*_{012,\cdots,i}$ and $C_{l'_{i+1}}$ as well as the two edges $(u_i,u_i^{j(i+1)})$ and $(v_i,v_i^{j(i+1)})$, we can construct a fault-free even cycle $C'_{01\cdots i(i+1)}=(v_i,P_{01\cdots i}[v_i,u_i], u_i,u_i^{j(i+1)},P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}],v_i^{j(i+1)},v_i)$ which contains e. Obviously, $l(C'_{01\cdots i(i+1)})=l(P_{01\cdots i}[v_i,u_i])+l(P_{i+1}[u_i^{j(i+1)}, v_i^{j(i+1)}])+2$ where $l(P_{01\cdots i}[v_i,u_i])=(i+1)\times d^k-2\sum_{a=0}^{i}f_a-1$ and $l(P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}])=d^k-2f_{i+1}-1$. Therefore, the cycle $C'_{01\cdots i(i+1)}$ is $((i+2)\times d^k-2\sum_{a=0}^{i}f_a)$-cycle and contains the edge e.

Let $C^*_{012,\cdots,i}$ $(i=2,4,\cdots,d-5,d-3)$ be a fault-free even $((i+1)\times d^k-2\sum_{a=0}^{i}f_a-1)$-cycle containing the edge e, $C^{*'}_{012,\cdots,i}$ be a fault-free $((i+1)\times d^k-2\sum_{a=0}^{i}f_a)$-cycle containing the edge e. One can observe that there are at least $\frac{1}{2}\times[(i+1)d^k-1]-\sum_{a=0}^{i}f_a-1$ disjoint edges such that each of them differs with e in the cycle $C^*_{012,\cdots,i}$. Since $k\geqslant 3$ and $\sum_{a=0}^{k+1}f_a\leqslant k-1$, $\frac{1}{2}\times[(i+1)d^k-1]-\sum_{a=0}^{i}f_a-i>\sum_{a=i+1}^{k+1}f_a$. Therefore, $C^*_{012,\cdots,i}$ ($C^{*'}_{012,\cdots,i}$) has an edge (u_i, v_i), $(u_i,v_i)\notin\{e,(u_1,v_1),\cdots,(u_{i-1},v_{i-1})\}$, $u_i^{j(m)}$ is a fault-free vertex in $Q_k^m(d)$, $v_i^{j(m)}$ is a fault-free vertex in $Q_k^m(d)$ $(m\in\{i+1,i+2,\cdots,d-1\})$, $h(u_i, u_i^{j(m)})=1$, $h(v_i,v_i^{j(m)})=1$. We may assume that $m=i+1$ ($m\neq i+1$ is similar), i. e., $u_i^{j(i+1)}$ is a fault-free vertex in $Q_k^{i+1}(d)$, $v_i^{j(i+1)}$ is a fault-free vertex in $Q_k^{i+1}(d)$. The cycle $C^*_{012,\cdots,i}$ ($C^{*'}_{012,\cdots,i}$) can be represented as $(u_i,v_i,P_{012\cdots i}[v_i,u_i],u_i)$ where e lies on the $P_{012\cdots i}[v_i,u_i]$.

Since $f_{i+1}\leqslant k-2$, by induction hypothesis, there are even cycles with lengths from 4 to $d^k-2f_{i+1}-1$ in $Q_k^{i+1}(d)$ that each cycle contains the edge $(u_i^{j(i+1)},v_i^{j(i+1)})$, and there is a (d^k-2f_{i+11})-cycle in $Q_k^{i+1}(d)$ that the cycle contains the edge $(u_i^{j(i+1)},v_i^{j(i+1)})$. Let $C_{l_{i+1}}=(v_i^{j(i+1)},u_i^{j(i+1)},P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}],v_i^{j(i+1)})$ be an even l_{i+1}-cycle containing the edge $(u_i^{j(i+1)},v_i^{j(i+1)})$ in $Q_k^{i+1}(d)$ where $4\leqslant l_{i+1}\leqslant d^k-2f_{i+1}-1$, $C_{l'_{i+1}}=(v_i^{j(i+1)},u_i^{j(i+1)}, P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}],v_i^{j(i+1)})$ be a (d^k-2f_{i+1})-cycle containing the edge $(u_i^{j(i+1)},v_i^{j(i+1)})$ in $Q_k^{i+1}(d)$. Merging the two cycles $C^*_{012,\cdots,i}$ and $C_{l_{i+1}}$ as well as the two edges $(u_i,u_i^{j(i+1)})$ and $(v_i,v_i^{j(i+1)})$, we can construct a fault-free even cycle $C_{01\cdots i(i+1)}=(v_i, P_{01\cdots i}[v_i,u_i],u_i,u_i^{j(i+1)},P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}], v_i^{j(i+1)}, v_i)$ which contains e. Obviously, $l(C_{01\cdots i(i+1)})=l(P_{01\cdots i}[v_i,u_i])+l(P_{i+1}[u_i^{j(i+1)}, v_i^{j(i+1)}]+2$ where $l(P_{01\cdots i}[v_i,u_i])=(i+1)\times d^k-2\sum_{a=0}^{i}f_a-2$, and $l(P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}])=1,3,\cdots,d^k-2f_{i+1}-2$. Therefore, the cycle $C_{01\cdots i(i+1)}$ is of length from $(i+1)\times d^k-2\sum_{a=0}^{i}f_a+1$ to $(i+2)\times d^k-2\sum_{a=0}^{i+1}f_a-2$ and contains the edge e. Merging the two cycles $C^{*'}_{012,\cdots,i}$ and $C_{l'_{i+1}}$ as well as the two edges $(u_i,u_i^{j(i+1)})$ and $(v_i,v_i^{j(i+1)})$, we can construct a fault-free even cycle $C'_{01\cdots i(i+1)}=(v_i,P_{01\cdots i}[v_i,u_i], u_i,u_i^{j(i+1)},P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}],v_i^{j(i+1)},v_i)$ which contains e. Obviously, $l(C'_{01\cdots i(i+1)})=l(P_{01\cdots i}[v_i, u_i])+l(P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}])+2$ where $l(P_{01\cdots i}[v_i,u_i])=(i+1)\times d^k-2\sum_{a=0}^{i}f_a-1$ and $l(P_{i+1}[u_i^{j(i+1)},v_i^{j(i+1)}])=d^k-2f_{i+1}-1$. Therefore, the cycle $C'_{01\cdots i(i+1)}$ is $((i+2)\times d^k-2\sum_{a=0}^{i+1}f_a)$-cycle and contains the edge e.

Case 2: $e\notin E(Q_k^0(d))\cup E(Q_k^1(d))\cup\cdots\cup E(Q_k^{d-1}(d))$, i. e., $u\in Q_k^l(d)$ $(l\in\{0,1,\cdots,d-1\})$, $v\in Q_k^m(d)$ $(m\in\{0,1,\cdots,d-1\})$, $l\neq m$, e is an edge of dimension j and $v=u^{j(a)}$ $(j\in\{1,2,\cdots,k+1\}$, $a\in\{0,1,\cdots,d-1\})$.

The proof of Case 2 is similar to the proof of

Case 2 of Theorem 3.

Applying Theorem 6, we have

Corollary 7 Let $n \geqslant 3$ be an integer. For any subset F of $V(Q_n(d))$ ($d \geqslant 3$, d is an odd number) with $|F| \leqslant n-2$, every vertex of $Q_n(d)-F$ lies on a fault-free cycle of every even length from 4 to $d^n-2|F|$. Moreover, every vertex of $Q_n(d)-F$ lies on a fault-free cycle of length $d^n-2|F|$.

References

[1] LI T K, TSAI C H, TAN J J M, et al. Bipanconnectivity and edge-fault-tolerant bipancyclicity of hypercubes [J]. Information Processing Letters, 2003, 87: 107-110.

[2] TSAI C H. Cycles embedding in hyperbes with node failures [J]. Information Processing Letters, 2007, 102(6): 242-246.

[3] XU J M. Combinatorial theory in networks [M]. Beijing: Science Press, 2013.

[4] HWANG S C, CHEN G H. Cycles in butterfly graphs [J]. Networks, 2000, 35(2): 161-171.

[5] SAAD Y, SCHULTZ M H. Topological properties of hypercubes [J]. IEEE Transactions on Computers, 1988, 37(7): 867-872.

[6] XU J M, DU Z Z, XU M. Edge-fault-tolerant edge-bipancyclicity of hypercubes [J]. Information Processing Letters, 2005, 96: 146-150.

[7] FU J S. Fault-tolerant cycle embedding in the hypercube [J]. Parallel Computing, 2003, 29(6): 821-832.

[8] STEWART I A, XIANG Y. Bipanconnectivity and bipancyclicity in k-ary n-cubes [J]. IEEE Transactions on Parallel and Distributed Systems, 2009, 20(1): 25-33.

[9] CHENG D Q, HAO R X, FENG Y Q. Vertex-fault-tolerant cycles embedding in balanced hypercubes [J]. Information Sciences, 2014, 288: 449-461.

[10] HAO R X, ZHANG R, FENG Y Q, et al. Hamiltonian cycle embedding for fault tolerance in balanced hypercubes [J]. Applied Mathematics and Computation, 2014, 244: 447-456.

有节点故障的 *d* 进制 *n* 维方的圈嵌入

李赵祥

(中央民族大学理学院,北京 100081)

摘要:互连网络的容错能力是并行计算中的一个关键问题,而 d 进制 n 维方(超方的一般形式)在计算机的互连网络中已得到广泛的应用。本文考虑有节点故障的 d 进制 n 维方的容错性。F 是 d 进制 n 维方 $Q_n(d)$ 中的错误点集($n \geqslant 3$),且 $|F| \leqslant n-2$,证明了 $Q_n(d)$ 的每个无故障的边和无故障的点存在于长从 4 到 $d^n-2|F|$ 的无故障偶圈中。而且,当 d 是奇数时,$Q_n(d)$ 的每个无故障的边和无故障的点存在于长为 $d^n-2|F|$ 的无故障圈中。

关键词:圈嵌入　超方　故障容错　互联网络　d 进制

责任编辑:米慧芝

微信公众号投稿更便捷

联系电话:0771-2503923

邮箱:gxkx@gxas.cn

投稿系统网址:http://gxkx.ijournal.cn/gxkx/ch