第 19 卷第 1 期　　广西科学院学报　　Vol. 19, No. 1
2003 年 2 月　　Journal of Guangxi Academy of Sciences　　February 2003

关于三阶变系数线性微分方程的解

General Solutions of the Third Order Variable Coefficients Linear Differential Equation

刘　琼
Liu Qiong

（钦州师范高等专科学校数学与计算机科学系　钦州　535000）
(Dept. of Math. and Comp. Sci., Guangxi Qinzhou Teachers College, Qinzhou, 535000)

摘要　通过变量变换，将变系数线性常微分方程化为常系数线性常微分方程，再利用常数变易法，给出一类三阶变系数非齐线性微分方程的通解.

关键词　三阶变系数线性微分方程　变量变换　常系数线性微分方程　通解

中图法分类号　O175.14　A

Abstract　The general solutions of the third order inhomogeneous linear differential equation with variable coefficients is given, by the method of the variable transformation.

Key words　third order variable coefficients linear differential equation, variable transformation, constants coefficients linear differential equation, general solution

对于一般三阶变系数非齐线性微分方程

$$X'''(t) + a_1(t)X''(t) + a_2(t)X'(t) + a_3(t)X(t) = f(t), \tag{1}$$

其中 $a_1(t), a_2(t), a_3(t)$ 及 $f(t)$ 都是所考虑区间的连续函数，没有普遍的解法，但是当方程(1)系数满足一定的条件时，这类方程的通解便可求得，而且不同于文献[1]，即为

$$X(t) = (c_1 + c_2t + c_3t^2)e^{-\frac{1}{3}\int a_1(t)dt} + \{\int \frac{t^2}{2}e^{\frac{1}{3}\int a_1(t)dt}f(t)dt - t\int te^{\frac{1}{3}\int a_1(t)dt}f(t)dt + \frac{t^2}{2}\int e^{\frac{1}{3}\int a_1(t)dt}f(t)dt\}e^{-\frac{1}{3}\int a_1(t)dt}. \tag{*}$$

1　定理及推论

定理　对于方程(1)，当系数满足

$a_2(t) = \frac{1}{3}a_1^2(t) + a_1'(t), a_3(t) = \frac{1}{27}a_1^3(t) + \frac{1}{3}a_1(t)a_1'(t) + \frac{1}{3}a_1''(t)$ 时，可化为常系数线性方程

$$V'''(t) + pV''(t) + \frac{p^2}{3}V'(t) + \frac{p^3}{27}V(t) = e^{-\frac{1}{3}\int[p - a_1(t)]dt}f(t), \tag{2}$$

其中 p 为常数.

2002-12-12 收稿，2003-02-11 修回.

证明　引入变量 $X(t):=e^{\frac{1}{3}\int[p-a_1(t)]dt}V(t)$（$p$ 为常数），则有

$$X'(t)=[\frac{p-a_1(t)}{3}V(t)+V'(t)]e^{\frac{1}{3}\int[p-a_1(t)]dt},$$

$$X''(t)=\{[\frac{1}{9}(p-a_1(t))^2-\frac{a_1'(t)}{3}]V(t)+\frac{2}{3}[p-a_1(t)]V'(t)+V''(t)\}e^{\frac{1}{3}\int[p-a_1(t)]dt},$$

$$X'''(t)=\{[\frac{1}{27}(p-a_1(t))^3-\frac{a_1'(t)}{3}(p-a_1(t))-\frac{a_1''(t)}{3}]V(t)+[\frac{1}{3}(p-a_1(t))^2-a_1'(t)]V'(t)+[p-a_1(t)]V''(t)+V'''(t)\}e^{\frac{1}{3}\int[p-a_1(t)]dt},$$

将 $X(t),X'(t),X''(t)$ 及 $X'''(t)$ 代入方程(1)得

$$V'''(t)+pV''(t)+[\frac{1}{3}(p-a_1(t))^2-a_1'(t)+\frac{2}{3}a_1(t)(p-a_1(t))+a_2(t)]V'(t)+[\frac{1}{27}(p-a_1(t))^3-\frac{a_1'(t)}{3}(p-a_1(t))-\frac{a_1''(t)}{3}+\frac{a_1(t)}{9}(p-a_1(t))^2-\frac{a_1(t)a_1'(t)}{3}+\frac{a_2(t)}{3}(p-a_1(t))+a_3(t)]V(t)=f(t)e^{-\frac{1}{3}\int[p-a_1(t)]dt}.$$

整理得

$$V'''(t)+pV''(t)+[\frac{p^2}{3}+a_2(t)-\frac{a_1^2(t)}{3}-a_1'(t)]V'(t)+\{\frac{p^3}{27}+a_3(t)-[\frac{1}{27}a_1^3(t)+\frac{1}{3}a_1(t)a_1'(t)+\frac{1}{3}a_1''(t)]\}V(t)=f(t)e^{-\frac{1}{3}\int[p-a_1(t)]dt}, \tag{3}$$

将 $a_2(t)=\frac{1}{3}a_1^2(t)+a_1'(t),a_3(t)=\frac{1}{27}a_1^3(t)+\frac{1}{3}a_1(t)a_1'(t)+\frac{1}{3}a_1''(t)$ 代入(3)即得方程(2).

定理证毕.

注：p 为任一常数. 当 $p=0$ 时，所作变换同文献[1]，$p\neq 0$ 时，表示变换不唯一.

要得到方程(1)的解，可先求方程(2)的通解，求方程(2)的通解，只要求出方程(2)对应的齐线性微分方程

$$V'''(t)+pV''(t)+\frac{p^2}{3}V'(t)+\frac{p^3}{27}V(t)=0 \tag{3'}$$

的通解，再利用常数变易法，便可求得方程(2)的通解.

很显然，方程(3′)的通解为

$V(t)=(c_1+c_2t+c_3t^2)e^{-\frac{p}{3}t}$，其中 c_1,c_2，及 c_3 均为任意常数.

令：$V(t)=[c_1(t)+tc_2(t)+t^2c_3(t)]e^{-\frac{p}{3}t}$ 是方程(2)的解，代入(2)有

$$\begin{cases}e^{-\frac{p}{3}t}c_1'(t)+te^{-\frac{p}{3}t}c_2'(t)+t^2e^{-\frac{p}{3}t}c_3'(t)=0,\\(e^{-\frac{p}{3}t})'c_1'(t)+(te^{-\frac{p}{3}t})'c_2'(t)+(t^2e^{-\frac{p}{3}t})'c_3'(t)=0,\\(e^{-\frac{p}{3}t})''c_1'(t)+(te^{-\frac{p}{3}t})''c_2'(t)+(t^2e^{-\frac{p}{3}t})''c_3'(t)=f(t)e^{-\frac{1}{3}\int[p-a_1(t)]dt}.\end{cases}$$

化简得

$$\begin{cases}c_1'(t)+tc_2'(t)+t^2c_3'(t)=0,\\-\frac{p}{3}c_1'(t)+(1-\frac{p}{3}t)c_2'(t)+(2t-\frac{p}{3}t^2)c_3'(t)=0,\\\frac{p^2}{9}c_1'(t)+(-\frac{2p}{3}+\frac{p^2}{9}t)c_2'(t)+(2-\frac{4p}{3}t+\frac{p^2}{9}t^2)c_3'(t)=f(t)e^{\frac{1}{3}\int a_1(t)dt}.\end{cases}$$

解得

$$c_1(t)=\frac{1}{2}\int t^2e^{\frac{1}{3}\int a_1(t)dt}f(t)dt+c_1,$$

$$c_2(t)=-\int te^{\frac{1}{3}\int a_1(t)dt}f(t)dt+c_2,$$

$$c_3(t)=\frac{1}{2}\int e^{\frac{1}{3}\int a_1(t)dt}f(t)dt+c_3,$$

则 $V(t)=(c_1+c_2t+c_3t^2)e^{-\frac{p}{3}t}+\{\frac{1}{2}\int t^2e^{\frac{1}{3}\int a_1(t)dt}f(t)dt-t\int te^{\frac{1}{3}\int a_1(t)dt}f(t)dt+\frac{t^2}{2}\int e^{\frac{1}{3}\int a_1(t)dt}f(t)dt\}e^{-\frac{p}{3}t}$

是方程(2)的解.

因此 式(*)是方程(1)的解.又由于c_1,c_2及c_3均为任意常数,是彼此独立的.故(*)也是方程(1)的通解.

推论 对方程(1),当系数满足定理条件及$f(t)=e^{-\frac{1}{3}\int a_1(t)dt}$时,方程(1)的通解为

$$X(t)=(c_1+c_2t+c_3t^2+\frac{1}{6}t^3)e^{-\frac{1}{3}\int a_1(t)dt}.$$

证明显而易见.

2 应用

例1 $X'''(t)+\frac{1}{t}X''(t)-\frac{2}{3t^2}X'(t)+\frac{10}{27t^3}X(t)=t^3.$

因为 $a_1(t)=\frac{1}{t},a_2(t)=-\frac{2}{3t^2},a_3(t)=\frac{10}{27t^3}$,显然满足定理的条件,故由定理得该方程的通解为

$$X(t)=(c_1+c_2t+c_3t^2)e^{-\frac{1}{3}\int\frac{1}{t}dt}+\{\frac{1}{2}\int t^2e^{\frac{1}{3}\int\frac{1}{t}dt}t^3dt-t\int te^{\frac{1}{3}\int\frac{1}{t}dt}t^3dt+\frac{t^2}{2}\int e^{\frac{1}{3}\int\frac{1}{t}dt}t^3dt\}e^{-\frac{1}{3}\int\frac{1}{t}dt}=$$
$$(c_1+c_2t+c_3t^2)t^{-\frac{1}{3}}+\frac{27t^6}{19\cdot 13\cdot 8\cdot 2}.$$

例2 $X'''+mt^3X''+(\frac{m^2}{3}t^6+3mt^2)X'(t)+(\frac{m^3}{27}t^9+m^2t^5+2mt)X(t)=e^{-\frac{mt^4}{12}}.$

显然,$a_1(t)=mt^3$(m为常数),$a_2(t)=\frac{1}{3}(mt^3)^2+(mt^3)'=\frac{1}{3}a_1^2(t)+a_1'(t)$,

$$a_3(t)=\frac{1}{27}(mt^3)^3+\frac{1}{3}(mt^3)(3mt^2)+2mt=\frac{1}{27}a_1^3(t)+\frac{1}{3}a_1(t)a_1'(t)+\frac{1}{3}a_1''(t),$$

而且$f(t)=e^{-\frac{m}{12}t^4}=e^{-\frac{1}{3}\int a_1(t)dt}$,满足推论条件,则由推论可得通解为

$$X(t)=(c_1+c_2t+c_3t^2+\frac{1}{6}t^3)e^{-\frac{mt^4}{12}}.$$

例3 $X'''(t)+\sin tX''(t)+(\frac{1}{3}\sin^2t+\cos t)X'(t)+(\frac{1}{27}\sin^3t+\frac{1}{6}\sin 2t-\frac{1}{3}\sin t)X(t)=e^{-\frac{1}{3}\cos t}.$

由推论得,它的通解为

$$X(t)=(c_1+c_2t+c_3t^2+\frac{1}{6}t^3)e^{\frac{1}{3}\cos t}.$$

致谢

衷心感谢导师罗桂烈教授的热心指导.

参考文献

1 汤光宋.常微分方程专题研究.武汉:华中理工大学出版社,1995.

(责任编辑:黎贞崇)