广西科学院学报 2008,24(2):89～91
Journal of Guangxi Academy of Sciences Vol. 24,No. 2 May 2008

关于四元数正则函数的两个充要条件*

Two Necessary and Sufficient Conditions of Quaternionic Regular Functions

周玉兴，黄敬频
ZHOU Yu-xing, HUANG Jing-pin

（广西民族大学数学与计算机科学学院，广西南宁 530006）
(Department of Mathematics and Computer Science, Guangxi University for Nationalities, Nanning, Guangxi, 530006, China)

摘要：运用已定义的正则函数，从外微分的角度，给出四元数函数左(右)正则的2个充要条件.

关键词：四元数 正则函数 外代数

中图法分类号：O151.23 **文献标识码**：A **文章编号**：1002-7378(2008)02-0089-03

Abstract: By introducing quaternion regular functions, two necessary and sufficient conditions of quaternionic Left-regular (or right-regular) functions in terms of exterior differential were given.

Key words: quaternion, regular functions, exterior algebra

四元数是四维可除代数，由爱尔兰数学家哈密顿于1843年首先发现.四元数是一种形如 $q = a_0 + a_1 i + a_2 j + a_3 k$ 的数，其中 $a_i \in \mathrm{R}(i = 0,1,2,3)$，1，$i,j,k$ 是基矢，且满足 $i^2 = j^2 = k^2 = ijk = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$.四元数的全体记为 H.由于实数域和复数域都是四元数体的子域，因此自然会考虑能否把实分析和复分析的一些重要结论推广到四元数体中去.20世纪30年代以前，不少数学工作者在这方面做过尝试，但由于他们定义的全纯函数有很大局限性，因此未能把复分析中的一些重要定理推广到四元数体上.直到1935年，R. Fueter定义了四元数正则函数，建立了四元数柯西-黎曼方程，人们才把复分析中的一些重要定理在四元数体上进一步拓广，如柯西定理、柯西积分公式和留数定理等[1~5].

文献[6]讨论四元数函数可微的充分条件和必要条件，文献[7]研究四元数正则函数收敛性和边界条件.但是，文献[6]研究四元数函数可微性，仅对形如 $f(q) = a + bq (a,b \in H)$ 的四元数函数适用，而且这样的函数是线性的.对四元数幂级数(例如 $f(q) = q^2$)是不成立的.文献[7]主要局限于研究有界区域四元数正则函数收敛性及其边界条件.本文主要运用已定义的正则函数和从外微分的角度，给出四元数正则函数的两个充要条件，改进并推广了文献[6]的相关结果.

1 相关定义

定义1 设 $f: \mathrm{R}^4 \to H$，且 D 是微分算子，

若 $fD = \frac{\partial f}{\partial a_0} + \frac{\partial f}{\partial a_1} i + \frac{\partial f}{\partial a_2} j + \frac{\partial f}{\partial a_3} k = 0$，则称 f 在 q 右正则；

若 $Df = \frac{\partial f}{\partial a_0} + i \frac{\partial f}{\partial a_1} + j \frac{\partial f}{\partial a_2} + k \frac{\partial f}{\partial a_3} = 0$，则称 f 在 q 左正则.

为方便简写，把四元数单位元 $1,i,j,k$ 对应记作 e_0, e_1, e_2, e_3，它们的运算法则是 $e_0 e_i = e_i e_0 = e_i (i = 0,1,2,3)$；$e_i e_j = -e_j e_i (i,j = 1,2,3)$；$(e_1)^2 = (e_2)^2 = (e_3)^2 = e_1 e_2 e_3 = -e_0$.所以，四元数 $q = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 = \sum_{i=0}^{3} a_i e_i$，其中 $a_i \in \mathrm{R}(i = 0,1,2,3)$.

对于 n 维实向量空间 V，若 $x, y \in V^1$ 则其外积满足 $x \wedge y = -y \wedge x, x \wedge x = 0$.由于四元数体可看作实四维向量空间 R^4，其外代数 $\wedge_{\mathrm{R}}$ 由 da_0, da_1，

收稿日期：2008-02-21
修回日期：2008-04-08

作者简介：周玉兴(1975-)，男，硕士研究生，主要从事四元数分析和泛函算子理论研究工作。

* 广西民族大学2007年研究生教育创新计划项目(gxun-chx0751)资助。

da_2, da_3 张成，故四元数体 H 的外代数 $\wedge_H$ 可由 $e_i da_j (i,j=0,1,2,3)$ 张成. 于是，四元数 q 的 1-形式 $\wedge_H^1$ 可表示成

$$dq = e_0 da_0 + e_1 da_1 + e_2 da_2 + e_3 da_3 = \sum_{i=0}^{3} e_i da_i.$$

类似地，

$$dq \wedge dq = e_1 da_1 \wedge da_3 + e_2 da_3 \wedge da_1 + e_3 da_1 \wedge da_2 \in \wedge_H^2.$$

定义 2 设有同构算子 $*: \wedge_H^1 \to \wedge_H^3$，则四元数 q 的外微分 $Dq = *(dq) \in \wedge_H^3$，即 Dq 可定义为

$$Dq = e_0 da_1 \wedge da_2 \wedge da_3 - e_1 da_0 \wedge da_2 \wedge da_3 + e_2 da_0 \wedge da_1 \wedge da_3 - e_3 da_0 \wedge da_1 \wedge da_2 \in \wedge_H^3.$$

定义 3[2] 设 $f: U(\text{开集}) \subset \mathrm{R}^4 \to H$ 是解析的，则 f 在 $q \in U$ 的微分定义为

$$df = \frac{\partial f}{\partial a_0} da_0 + \frac{\partial f}{\partial a_1} da_1 + \frac{\partial f}{\partial a_2} da_2 + \frac{\partial f}{\partial a_3} da_3.$$

2 主要结果

由文献[8]，对于区域 Ω 中的解析函数 $f(z) = u(z) + iv(z)(z = x + iy \in \Omega)$，有

$\frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}), \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})$，其中 $\bar{z} = x - iy$.

类似复分析，在四元数体上引入微分算子方程

$$\bar{\partial}_l f = \frac{1}{2}(\frac{\partial f}{\partial a_0} + i\frac{\partial f}{\partial a_1} + j\frac{\partial f}{\partial a_2} + k\frac{\partial f}{\partial a_3}); \partial_l f = \frac{1}{2}(\frac{\partial f}{\partial a_0} - i\frac{\partial f}{\partial a_1} - j\frac{\partial f}{\partial a_2} - k\frac{\partial f}{\partial a_3});$$

$$\bar{\partial}_r f = \frac{1}{2}(\frac{\partial f}{\partial a_0} + \frac{\partial f}{\partial a_1}i + \frac{\partial f}{\partial a_2}j + \frac{\partial f}{\partial a_3}k); \partial_r f = \frac{1}{2}(\frac{\partial f}{\partial a_0} - \frac{\partial f}{\partial a_1}i - \frac{\partial f}{\partial a_2}j - \frac{\partial f}{\partial a_3}k).$$

命题 1 设 $f: \mathrm{R}^4 \to H$，则 f 是左（或右）正则的充要条件是 $\bar{\partial}_l f = 0$（或 $\bar{\partial}_r f = 0$）.

由于四元数可用复数对 $q = z_1 + z_2 j$ 来表示，其中 $(z_1, z_2) = (a_0 + a_1 i, a_2 + a_3 i)$，所以 H 同构于 C^2. 易知，$\bar{q} = \bar{z}_1 - z_2 j$. 同理，$q = z_1 + j z_2$，其中 $(z_1, z_2) = (a_0 + i a_1, a_2 - i a_3)$. 对任意 $a \in C$，显然有 $aj = j\bar{a}$. 于是可设四元数函数 $f(q) = f_1(z_1, z_2) + f_2(z_1, z_2)j$.

定理 1 设四元数函数 $f(q) = f_1(z_1, z_2) + f_2(z_1, z_2)j$，则

(1) $f(q)$ 是左正则的充要条件是 $f_1(z_1, z_2)$，$f_2(z_1, z_2)$，满足条件：

$$\frac{\partial f_1(z_1,z_2)}{\partial \bar{z}_1} = \frac{\partial \overline{f_2(z_1,z_2)}}{\partial \bar{z}_2}; \frac{\partial f_1(z_1,z_2)}{\partial z_2} = -\frac{\partial \overline{f_2(z_1,z_2)}}{\partial z_1}. \quad (2.1)$$

(2) $f(q)$ 是右正则的充要条件是 $f_1(z_1, z_2)$，$f_2(z_1, z_2)$ 满足条件：

$$\frac{\partial f_1(z_1,z_2)}{\partial \bar{z}_1} = \frac{\partial \overline{f_2(z_1,z_2)}}{\partial \bar{z}_2}; \frac{\partial f_2(z_1,z_2)}{\partial \bar{z}_1} = -\frac{\partial \overline{f_1(z_1,z_2)}}{\partial \bar{z}_2}. \quad (2.2)$$

证明 (1) 利用四元数体上的左微分算子可得

$$\frac{\partial_l}{\partial \bar{q}} = \frac{\partial}{\partial a_0} + i\frac{\partial}{\partial a_1} + j\frac{\partial}{\partial a_2} + k\frac{\partial}{\partial a_3} = (\frac{\partial}{\partial a_0} + i\frac{\partial}{\partial a_1}) + j(\frac{\partial}{\partial a_2} - i\frac{\partial}{\partial a_3}) = 2(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial z_2}).$$

当 f 左正则时，$\frac{\partial_l}{\partial \bar{q}} f(q) = 0$，可得 $2(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial z_2})(f_1(z_1,z_2) + f_2(z_1,z_2)j) = 0$，

于是

$$(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial z_2})(f_1(z_1,z_2) + j\overline{f_2(z_1,z_2)}) = 0,$$

所以

$$\frac{\partial f_1(z_1,z_2)}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{z}_1}(j\overline{f_2(z_1,z_2)}) + j\frac{\partial}{\partial z_2}f_1(z_1,z_2) + (j\frac{\partial}{\partial z_2})\cdot(j\overline{f_2(z_1,z_2)}) = \frac{\partial f_1(z_1,z_2)}{\partial \bar{z}_1} + j\frac{\partial}{\partial z_1}(\overline{f_2(z_1,z_2)}) + j\frac{\partial}{\partial z_2}f_1(z_1,z_2) + (\frac{\partial}{\partial \bar{z}_2}j)\cdot(j\overline{f_2(z_1,z_2)}) = (\frac{\partial f_1(z_1,z_2)}{\partial \bar{z}_1} - \frac{\partial \overline{f_2(z_1,z_2)}}{\partial \bar{z}_2}) + j(\frac{\partial \overline{f_2(z_1,z_2)}}{\partial z_1} + \frac{\partial f_2(z_1,z_2)}{\partial z_2}) = 0,$$

从而

$$\frac{\partial f_1(z_1,z_2)}{\partial \bar{z}_1} = \frac{\partial \overline{f_2(z_1,z_2)}}{\partial \bar{z}_2}; \frac{\partial \overline{f_2(z_1,z_2)}}{\partial z_1} = -\frac{\partial f_1(z_1,z_2)}{\partial z_2}.$$

显然，上述过程是可逆的，因此，当条件 (2.1) 满足时，$f(q)$ 是左正则的.

(2) 利用四元数体上的右微分算子可得

$$\frac{\partial_r}{\partial \bar{q}} = \frac{\partial}{\partial a_0} + \frac{\partial}{\partial a_1}i + \frac{\partial}{\partial a_2}j + \frac{\partial}{\partial a_3}k = (\frac{\partial}{\partial a_0} + \frac{\partial}{\partial a_1}i) + (\frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3}i)j = 2(\frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{z}_2}j).$$

当 f 右正则时，$\frac{\partial_r}{\partial \bar{q}} f(q) = 0$，可得 $2(\frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{z}_2}j)(f_1(z_1,z_2) + f_2(z_1,z_2)j) = 0$.

于是

$$\frac{\partial f_1(z_1,z_2)}{\partial \bar{z}_1} + \frac{\partial f_2(z_1,z_2)}{\partial \bar{z}_1}j + \frac{\partial}{\partial \bar{z}_2}(j\cdot$$

$$f_1(z_1,z_2)) + (\frac{\partial}{\partial \bar{z}_2} j)\cdot(f_2(z_1,z_2)j) = \frac{\partial f_1(z_1,z_2)}{\partial \bar{z}_1} + \frac{\partial f_2(z_1,z_2)}{\partial \bar{z}_1} j + \frac{\partial}{\partial \bar{z}_2}(\overline{f_1(z_1,z_2)} j) + (\frac{\partial}{\partial \bar{z}_2} j)\cdot (j\ \overline{f_2(z_1,z_2)}) = \frac{\partial f_1(z_1,z_2)}{\partial \bar{z}_1} + \frac{\partial f_2(z_1,z_2)}{\partial \bar{z}_1} j + \frac{\partial \overline{f_1(z_1,z_2)}}{\partial \bar{z}_2} j - \frac{\partial \overline{f_2(z_1,z_2)}}{\partial \bar{z}_2} = (\frac{\partial f_1(z_1,z_2)}{\partial \bar{z}_1} - \frac{\partial \overline{f_2(z_1,z_2)}}{\partial \bar{z}_2}) + (\frac{\partial f_2(z_1,z_2)}{\partial \bar{z}_1} + \frac{\partial \overline{f_1(z_1,z_2)}}{\partial \bar{z}_2}) j = 0,$$

从而

$$\frac{\partial f_1(z_1,z_2)}{\partial \bar{z}_1} = \frac{\partial \overline{f_2(z_1,z_2)}}{\partial \bar{z}_2};\ \frac{\partial f_2(z_1,z_2)}{\partial \bar{z}_1} = -\frac{\partial \overline{f_1(z_1,z_2)}}{\partial \bar{z}_2}.$$

显然,上述过程亦可逆,因此,当条件(2.2)满足时,$f(q)$ 是右正则的.

定理 2 设 $f:U \to H$,其中 $U \subset \mathrm{R}^4$,$f \in C^1$ 则

(1)f 是左正则的当且仅当 $Dq \wedge df = 0$;

(2)f 是右正则的当且仅当 $df \wedge Dq = 0$.

证明 (1)因为 $da_i \wedge da_i = 0$,$da_i \wedge da_j = -da_j \wedge da_i$,故由外微分及 df 的定义可得

$$Dq \wedge df = (e_0 da_1 \wedge da_2 \wedge da_3 - e_1 da_0 \wedge da_2 \wedge da_3 + e_2 da_0 \wedge da_1 \wedge da_3 - e_3 da_0 \wedge da_1 \wedge da_2) \wedge (\frac{\partial f}{\partial a_0} da_0 + \frac{\partial f}{\partial a_1} da_1 + \frac{\partial f}{\partial a_2} da_2 + \frac{\partial f}{\partial a_3} da_3) = (e_0 \frac{\partial f}{\partial a_0} da_1 \wedge da_2 \wedge da_3 \wedge da_0 + e_0 \frac{\partial f}{\partial a_1} da_1 \wedge da_2 \wedge da_3 \wedge da_1 + e_0 \frac{\partial f}{\partial a_2} da_1 \wedge da_2 \wedge da_3 \wedge da_2 + e_0 \frac{\partial f}{\partial a_3} da_1 \wedge da_2 \wedge da_3 \wedge da_3) - (e_1 \frac{\partial f}{\partial a_0} da_0 \wedge da_2 \wedge da_3 \wedge da_0 + e_1 \frac{\partial f}{\partial a_1} da_0 \wedge da_2 \wedge da_3 \wedge da_1 + e_1 \frac{\partial f}{\partial a_2} da_0 \wedge da_2 \wedge da_3 \wedge da_2 + e_1 \frac{\partial f}{\partial a_3} da_0 \wedge da_2 \wedge da_3 \wedge da_3) + (e_2 \frac{\partial f}{\partial a_0} da_0 \wedge da_1 \wedge da_3 \wedge da_0 + e_2 \frac{\partial f}{\partial a_1} da_0 \wedge da_1 \wedge da_3 \wedge da_1 + e_2 \frac{\partial f}{\partial a_2} da_0 \wedge da_1 \wedge da_3 \wedge da_2 + e_2 \frac{\partial f}{\partial a_3} da_0 \wedge da_1 \wedge da_3 \wedge da_3) - (e_3 \frac{\partial f}{\partial a_0} da_0 \wedge da_1 \wedge da_2 \wedge da_0 + e_3 \frac{\partial f}{\partial a_1} da_0 \wedge da_1 \wedge da_2 \wedge da_1 + e_3 \frac{\partial f}{\partial a_2} da_0 \wedge da_1 \wedge da_2 \wedge da_2 + e_3 \frac{\partial f}{\partial a_3} da_0 \wedge da_1 \wedge da_2 \wedge da_3) = e_0 \frac{\partial f}{\partial a_0} da_1 \wedge da_2 \wedge da_3 \wedge da_0 - e_1 \frac{\partial f}{\partial a_1} da_0 \wedge da_2 \wedge da_3 \wedge da_1 + e_2 \frac{\partial f}{\partial a_2} da_0 \wedge da_1 \wedge da_3 \wedge da_2 - e_3 \frac{\partial f}{\partial a_3} da_0 \wedge da_1 \wedge da_2 \wedge da_3 = -(e_0 \frac{\partial f}{\partial a_0} da_0 \wedge da_1 \wedge da_2 \wedge da_3 + e_1 \frac{\partial f}{\partial a_1} da_0 \wedge da_1 \wedge da_2 \wedge da_3 + e_2 \frac{\partial f}{\partial a_2} da_0 \wedge da_1 \wedge da_2 \wedge da_3 + e_3 \frac{\partial f}{\partial a_3} da_0 \wedge da_1 \wedge da_2 \wedge da_3) = -(e_0 \frac{\partial f}{\partial a_0} + e_1 \frac{\partial f}{\partial a_1} + e_2 \frac{\partial f}{\partial a_2} + e_3 \frac{\partial f}{\partial a_3}) da_0 \wedge da_1 \wedge da_2 \wedge da_3.$$

所以,f 是左正则的 $\Leftrightarrow Dq \wedge df = 0$.

(2)由外微分及 df 的定义可得

$$df \wedge Dq = (\frac{\partial f}{\partial a_0} da_0 + \frac{\partial f}{\partial a_1} da_1 + \frac{\partial f}{\partial a_2} da_2 + \frac{\partial f}{\partial a_3} da_3) \wedge (e_0 da_1 \wedge da_2 \wedge da_3 - e_1 da_0 \wedge da_2 \wedge da_3 + e_2 da_0 \wedge da_1 \wedge da_3 - e_3 da_0 \wedge da_1 \wedge da_2 = \frac{\partial f}{\partial a_0} e_0 da_0 \wedge da_1 \wedge da_2 \wedge da_3 - \frac{\partial f}{\partial a_1} e_1 da_1 \wedge da_0 \wedge da_2 \wedge da_3 + \frac{\partial f}{\partial a_2} e_2 da_2 \wedge da_0 \wedge da_1 \wedge da_3 - \frac{\partial f}{\partial a_3} e_3 da_3 \wedge da_0 \wedge da_1 \wedge da_2 = (\frac{\partial f}{\partial a_0} e_0 + \frac{\partial f}{\partial a_1} e_1 + \frac{\partial f}{\partial a_2} e_2 + \frac{\partial f}{\partial a_3} e_3) da_0 \wedge da_1 \wedge da_2 \wedge da_3.$$

所以,f 是右正则的 $\Leftrightarrow df \wedge Dq = 0$.

参考文献:

[1] Deavours C A. The Quaternion calculus[J]. Amer Math Mon,1973,80:995-1008.

[2] Sudbery. Quaternionic analysis[J]. Mat Proc Camb Phil Soc,1979, 85:199-225.

[3] 杨丕文. 四元数空间中的正则函数及其某些边值问题[J]. 四川师范大学学报:自然科学版,1994,17(6):1-7.

[4] 乔玉英. 四元数广义正则函数的斜微商边值问题[J]. 数学物理学报,1997, 17(4): 447-451.

[5] 杨丕文. 四元数分析中的T算子与两类边值问题[J]. 数学学报,2001,44(2):343-350.

[6] 王秋媛. 四元数的解析性[J]. 北方交通大学学报,2000, 24(2):23-26

[7] 邹洁. 关于四元数正则函数[J]. 工科数学,1999,15(4):74-79.

[8] Ahlfors L V. Comples analysis [M]. Third Edition. Beijing:China Machine Press,2004.

(责任编辑:尹 闯)