On the Equivalence of Crossed Coproducts*
关于交叉余积的等价

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Abstract: The dual definition of crossed product is introduced. The isomorphism of special case between crossed coproducts coalgebra and twist coproduct coalgebra is considered. The question when two crossed coproducts are equivalence as coalgebra is discussed.

Key words: coalgebra, crossed coproducts, equivalence, inner coaction

摘要:引进交叉积的对偶定义交叉余积,并在交叉余积余代数和特殊扭余积余代数同构时,讨论交叉余积成为余代数的等价问题.

关键词:余代数 交叉余积 等价 内余作用

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S. Montgomery[1] introduced the definition of crossed product and discuss its properties. Wang Shuan-hong[2] discuss the conditions of crossed coproduct as a coalgebra. This article introduced the dual definition of crossed product. In this paper we consider the isomorphism of special case between crossed coproducts coalgebra and twist coproduct coalgebra. The question about when two crossed coproducts are equivalence as coalgebra is discussed. Some terms which are not defined here, please refer to reference[3].

1 Definitions

Let H be a Hopf algebra and C a coalgebra, K be a field. H and C are assumed to be over K unless stated otherwise.

Definition 1 Assume C is a weak left H-comodule that α is a linear map $\alpha: C \to H \otimes H$, $\alpha(c) = \Sigma\alpha_1(c) \otimes \alpha_2(c), \forall c \in C$. As a vector space $C \times_\alpha H = C \otimes H$ with comultiplication $\triangle$, $\triangle(c \times h) = \Sigma c_1 \times c_2^1\alpha_1(c_3)h_1 \otimes c_2^2 \times \alpha_2(c_3)h_2$, $\rho(c) = \Sigma c^1 \otimes c^2$ is the left H- comodule structure map, $\forall c \in C, h \in H$. Here we written $c \times h$ for the tensor $c \otimes h$. $C \times_\alpha H$ is a crossed coproduct using ρ and α, if $\varepsilon(c \times h) = \varepsilon_C(c)\varepsilon_H(h)$ as its counit and coassociativity are satisfied.

Definition 2 Let $u \in \mathrm{Hom}(C,H)$ is invertible, then definition $\alpha \in \mathrm{Hom}(C, H \otimes H)$, $\alpha(c) = \Sigma u^{-1}(c_2)u(c_3)_1 \otimes u^{-1}(c_1)u(c_3)_2, \forall c \in C$.

Definition 3 Let C is a weak left H- comodule coalgebra, then coaction is inner, if there exist some invertibles $u \in \mathrm{Hom}(C,H)$, such that comodule structure map is $\rho_u(c) = \Sigma u(c_1)u^{-1}(c_3) \otimes c_2, \forall c \in C$.

2 Main Results

Now we consider when two crossed coproducts are equivalence. Before discuss the general case of it, we consider the special case of inner coaction.

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Let $u \in \mathrm{Hom}(C,H)$ is invertible and $\varepsilon_H u = \varepsilon_C$, $\rho_u(c) = \Sigma u(c_1)u^{-1}(c_3) \otimes c_2$, $\rho_1(c) = 1 \otimes c$ are two left H- comodule structure maps. $\alpha, \beta: C \to H \otimes H$, such that

$$\alpha(c) = \Sigma(u^{-1}(c_2) \otimes u^{-1}(c_1))\beta(c_3)\triangle(u(c_4)),$$

$\forall c \in C$.

Since u is invertible. We have

$$\beta(c) = \Sigma(u(c_1) \otimes u(c_2))\alpha(c_3)\triangle(u^{-1}(c_4)),$$

$\forall c \in C$.

Remark 1 Following the definition, we know that $C \times_\beta H$ is a crossed coproduct using ρ_u and β; similarly, $C \times_\alpha H$ is also a crossed coproduct using ρ_1 and α. Here we write $C_\alpha[H]$ instead of $C \times_\alpha H$.

Proposition 1 $C \times_\beta H$ be a crossed coproduct using ρ_u and β, such that coaction of H on C is inner, via some invertible $u \in \mathrm{Hom}(C,H)$. Define $\alpha \in \mathrm{Hom}(C, H \otimes H)$ by $\alpha(c) = \Sigma(u^{-1}(c_2) \otimes u^{-1}(c_1))\beta(c_3)\triangle(u(c_4)), \forall c \in C$. Then $C \times_\beta H \cong C_\alpha[H]$, a crossed coproduct using ρ_1 and α, via a coalgebra isomorphism which is also a left C-comodule, right H- module map.

Proof Define $\psi: C \times_\beta H \to C_\alpha[H], \psi(c \otimes h) = \Sigma c_1 \otimes u^{-1}(c_2)h, \varphi: C_\alpha[H] \to C \times_\beta H, \varphi(c \otimes h) = \Sigma c_1 \otimes u(c_2)h$, then φ is the inverse of ψ and it is straightforward to check that. Now we check that ψ is a coalgebra map.

$\forall c \in C, \forall h \in H$. We have

$\triangle_\alpha\psi(c \otimes h) = \Sigma c_1 \otimes \alpha_1(c_3)(u^{-1}(c_4)h)_1 \otimes c_2 \otimes \alpha_2(c_3)(u^{-1}(c_4)h)_2 = \Sigma c_1 \otimes u^{-1}(c_4)\beta_1(c_5)u(c_6)u^{-1}(c_7)_1 h_1 \otimes c_2 \otimes u^{-1}(c_3)\beta_2(c_5)u(c_6)u^{-1}(c_7)_2 h_2 = \Sigma c_1 \otimes u^{-1}(c_4)\beta_1(c_5)h_1 \otimes c_2 \otimes u^{-1}(c_3)\beta_2(c_5)h_2$.

$(\psi \otimes \psi)\triangle_\beta(c \otimes h) = (\psi \otimes \psi)(\Sigma c_1 \otimes u(c_2)u^{-1}(c_4)\beta_1(c_5)h_1 \otimes c_3 \otimes \beta_2(c_5)h_2) = (\Sigma c_1 \otimes u^{-1}(c_{11})u(c_2)u^{-1}(c_4)\beta_1(c_5)h_1 \otimes c_3 \otimes u^{-1}(c_{31})\beta_2(c_5)h_2 = \Sigma c_1 \otimes u^{-1}(c_4)\beta_1(c_5)h_1 \otimes c_2 \otimes u^{-1}(c_3)\beta_2(c_5)h_2$.

So $\triangle_\alpha\psi = (\psi \otimes \psi)\triangle_\beta$. Then $\varepsilon_\beta(c \otimes h) = \varepsilon_C(c)\varepsilon_H(h)$.

$\varepsilon_\alpha\psi(c \otimes h) = \varepsilon_\alpha(\Sigma c_1 \otimes u^{-1}(c_2)h) = \Sigma\varepsilon_C(c_1)\varepsilon_H(u^{-1}(c_2)h) = \Sigma\varepsilon_C(c_1)\varepsilon_H(u^{-1}(c_2))\varepsilon_H(h) = \varepsilon_C(c_1)\varepsilon_C(c_2)\varepsilon_H(h) = \varepsilon_C(c_1\varepsilon_C(c_2))\varepsilon_H(h) = \varepsilon_C(c)\varepsilon_H(h)$.

So $\varepsilon_\beta = \varepsilon_\alpha\psi$.

Next, we check that ψ is a left C- comodule map.

$(\triangle_C \otimes I)\psi(c \otimes h) = (\triangle_C \otimes I)(\Sigma c_1 \otimes u(c_2)h) = \Sigma c_1 \otimes c_2 \otimes u(c_3)h$,

$(I \otimes \psi)(\triangle_C \otimes I)(c \otimes h) = (I \otimes \psi)(\Sigma c_1 \otimes c_2 \otimes h) = \Sigma c_1 \otimes c_2 \otimes u(c_3)h$.

So $(\triangle_C \otimes I)\psi = (I \otimes \psi)(\triangle_C \otimes I)$.

Also, ψ is a right H- module map.

$(I \otimes M_H)(\psi \otimes I)(c \otimes h \otimes l) = (I \otimes M_H) \cdot (\Sigma c_1 \otimes u(c_2)h \otimes l) = \Sigma c_1 \otimes u(c_2)hl$,

$\psi(I \otimes M_H)(c \otimes h \otimes l) = \psi(c \otimes hl) = \Sigma c_1 \otimes u(c_2)hl$.

So $(I \otimes M_H)(\psi \otimes I) = \psi(I \otimes M_H)$.

Example 1 Let $C \times_\alpha H$ be acrossed coproduct, such that H- coaction is inner via some $u \in \mathrm{Coalg}(C, H)$. In this case we say that the coaction is strongly inner. Then $C \times_\alpha H \cong C \otimes H$, for α becomes trivial.

The converse of proposition 1 is also true. That is, if $C \times_\beta H \cong C_\alpha[H]$, via a coalgebra isomorphism which is also a left C- comodule, right H- module map, then the original coaction must have been inner, via some invertible $u \in \mathrm{Hom}(C,H)$ such that ψ is given as in proposition 1.

Generally, one can give necessary and sufficient conditions for two crossed coproducts to be isomorphic. Then we have the following theorem.

Theorem 1 Let H be a Hopf algebra and C a coalgebra. Two linear map $\alpha, \alpha': C \to H \otimes H$. Assume that $\psi: C \times_\alpha H \to C \times'_\alpha H$ is a coalgebra isomorphism, which is also a left C- comodule, right H- module map. There exists an invertible map $u \in \mathrm{Hom}(C,H)$ such that

(1) $\psi(c \times h) = \Sigma c_1 \times' u^{-1}(c_2)h$.

(2) $\alpha'(c) = \Sigma(u^{-1}(c_1) \otimes u^{-1}(c_2))\alpha(c_3)\triangle(u(c_4))$.

(3) $\triangle_{\alpha'}(c \times' h) = \Sigma c_1 \times u^{-1}(c_2)c_3^1\alpha_1(c_4)(u(c_5)h)_1 \otimes (c_3^2)_1 \times u^{-1}(c_3^2)_2\alpha_2(c_4)(u(c_5)h)_2$.

Conversely, a map $u \in \mathrm{Hom}(C,H)$ such that (2) and (3) hold is given, then the map ψ in (1) is an isomorphism.

Proof Define $v \in \mathrm{Hom}(C,H)$ by $v(c) = (\varepsilon \otimes I)\psi(c \times 1), \forall c \in C$. So $(\varepsilon \otimes I)\psi(c \times h) = (\varepsilon \otimes I)\psi(c \times 1_H)h = v(c)h$. Since ψ is a right H- module, left C- comodule map, we have $(I \otimes \psi)(\triangle \otimes I) = (\triangle \otimes I)\psi$. $I \otimes \varepsilon \otimes I$ is applied to both side of equation. The right side becomes ψ ,and the left side becomes $(I \otimes (\varepsilon \otimes I)\psi)(\triangle \otimes I)$ which evaluated at $c \times k$ is

$$\psi(c \times h) = (I \otimes (\varepsilon \otimes I)\psi)(\triangle \otimes I)(c \times h) = (I \otimes (\varepsilon \otimes I)\psi)(\triangle(c) \times h) = \Sigma c_1 \otimes (\varepsilon \otimes I)\psi(c_2 \times h) = \Sigma c_1 \times' v(c_2)h.$$

As $\psi^{-1}: C \times'_\alpha H \to C \times_\alpha H$ is an isomorphism, satisfying the same hypotheses as ψ , we may set $u(c) = (\varepsilon \otimes I)\psi^{-1}(c \times 1)$ and conclude as above that $\psi^{-1}(c \times' h) = \Sigma c_1 \times u(c_2)h$. We claim that $v = u^{-1}$, for $c \times' 1_H = \psi^{-1}\psi(c \times' 1_H) = \psi^{-1}(\Sigma c_1 \times' v(c_2)1_H) = \Sigma c_1 \times' u(c_2)v(c_3)1_H$. $\varepsilon \otimes I$ to both sides, we see that $\Sigma u(c_1)v(c_2) = \varepsilon(c)1_H$. So $v = u^{-1}$. From the above, this proves(1).

Now the equation $\triangle_\alpha\psi^{-1}(c \times' h) = (\psi^{-1} \otimes \psi^{-1})\triangle_{\alpha'}(c \times' h)$ becomes

$$\Sigma c_1 \times c_2^1\alpha_1(c_3)(u(c_4)h)_1 \otimes c_2^2 \times \alpha_2(c_3)(u(c_4)h)_2 = \Sigma c_1 \times u(c_2)c_3^1\alpha'_1(c_4)h_1 \otimes (c_3^2)_1 \times u((c_3^2)_2)\alpha'_2(c_4)h_2.$$

Applying $\varepsilon \otimes I \otimes \varepsilon \otimes I$ to both sides, we see that

$$\Sigma\varepsilon(c_1) \times c_2^1\alpha_1(c_3)(u(c_4)h)_1 \otimes \varepsilon(c_2^2) \times \alpha_2(c_3)(u(c_4)h)_2 = \Sigma\varepsilon(c_1) \times u(c_2)c_3^1\alpha'_1(c_4)h_1 \otimes \varepsilon((c_3^2)_1) \times u((c_3^2)_2)\alpha'_2(c_4)h_2,$$

then

$$\Sigma c_1\alpha_1(c_2)(u(c_3)h)_1 \otimes \alpha_2(c_2)(u(c_3)h)_2 = \Sigma u(c_1)c_2^1\alpha'_1(c_3)h_1 \otimes u(c_2^2)\alpha'_2(c_3)h_2,$$

$$\Sigma(c_1 \otimes I)(\alpha_1(c_2) \otimes \alpha_2(c_2))\triangle(u(c_3)h) = \Sigma(u(c_1)c_2^1 \otimes u(c_2^2))(\alpha'_1(c_3) \otimes \alpha'_2(c_3))\triangle(h),$$

$$\Sigma(c_1 \otimes I)(\alpha(c_2))\triangle(u(c_3)h) = \Sigma(u(c_1)c_2^1 \otimes u(c_2^2))(\alpha'(c_3))\triangle(h).$$

Let $\rho_1(c) = 1 \otimes c, h = 1$,then we have

$$\Sigma (\alpha(c_1))\triangle(u(c_2)) = \Sigma(u(c_1) \otimes u(c_2))(\alpha'(c_3)),$$

So $\alpha'(c) = \Sigma(u^{-1}(c_1) \otimes u^{-1}(c_2))\alpha(c_3)\triangle(u(c_4)))$. This gives (2).

Now using the equation $\triangle_\alpha\psi^{-1}(c \times' h) = (\psi^{-1} \otimes \psi^{-1})\triangle_{\alpha'}(c \times' h)$ again and applying $(\psi \otimes \psi)$ to both sides, we see that the right side becomes $\triangle_{\alpha'}(c \times' h)$, and the left side becomes $(\psi \otimes \psi)\triangle_\alpha\psi^{-1}(c \times' h)$. From the above, this proves (3).

Corollary 1 With the same conditions of the theorem 1, if H is commutative, then (3) becomes $\triangle_{\alpha'}(c \times' h) = (I \times u^{-1}(c_1) \otimes I \times u^{-1}(c_2^2)_2)\triangle_\alpha(c_3 \times h)(I \times u(c_4)_1 \otimes I \times u(c_4)_2)$.

Proof H is commutative, then (3) becomes

$$\triangle_{\alpha'}(c \times' h) = \Sigma(I \times u^{-1}(c_1) \otimes I \times u^{-1}(c_2^2)_2)(c_1 \times c_2^1\alpha_1(c_3)h_1 \otimes (c_2^2)_1 \times \alpha_2(c_3)h_2)(I \times u(c_3)_1 \otimes I \times u(c_3)_2) = (I \times u^{-1}(c_1) \otimes I \times u^{-1}(c_2^2)_2)\triangle_\alpha(c_3 \times h)(I \times u(c_4)_1 \otimes I \times u(c_4)_2).$$

The result follows.

This corollary tells us about the relationship between $\triangle_{\alpha'}(c \times' h)$ and $\triangle_\alpha(c \times h)$, when H is commutative.

Remark 2 Let H be a Hopf algebra and C a coalgebra. Two crossed coproducts $C \times_\alpha H$ and $C \times'_\alpha H$ are equivalent if there exists a coalgebra isomorphism $\psi : C \times_\alpha H \to C \times'_\alpha H$ which is a left C-comodule morphism and right H- module morphism.

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